Isolation of the Conceptual Ingredients of Quantum Theory by Toy Theory Comparison

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Abstract

Quantum Theory is a rich and successful theory which suffers from a lack of clear and unanimous interpretation. I suggest a research programme that examines various toy theories that approximate some of the interesting features of quantum mechanics whilst being far easier to interpret. By examining these toy theories using category theory, the relevant structures and principles of the toy theory which lead to "quantum like" phenomena can be highlighted. In this way we can hope to better understand which axioms of Quantum Theory lead to which Quantum phenomena, and how. As a case study I examine Robert W. Spekkens's toy theory of epistemic states and compare it with Stabilizer Quantum Mechanics, noting the similarities and differences and their bearing on interpretations of Quantum Theory. I discuss an example of similarity with the toy theory, noting that a purely information theoretic principle on the states of the toy theory can give rise to a no cloning theorem for such states, and this suggests that a similar conceptual ingredient exists in Quantum Theory. I also discuss an example of contrast with the same toy theory: the existence or non-existence of a hidden variable interpretation. The comparison is made more precise by using Category Theory to describe the toy theory and quantum theory in a unified framework. The category theoretic ingredient of the phase group is shown to be the root of the presence or absence of non-locality in the theories. A four element cyclic phase group will generate non-locality in a theory whereas the four element Klein group fails to do so.

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1 Introduction

1.1 Inconsistency and Indefiniteness

At the turn of the 20th Century The Photoelectric Effect and The Double Slit Experiment had produced some unexpected results. The results were at first considered anomalous, but with increasing frequency they were reproduced and confirmed across the world and gathered enough attention to begin to overthrow the received Classical Physics of the day. The founders of Quantum Theory rapidly formulated radical and unintuitive notions of a *wave-particle duality*, of a fundamental *uncertainty* in parts of reality, and other exotic ideas [11] as conceptual ingredients of a new Physics. To some, Quantum Theory was born with a plethora of new and exciting concepts which captured the essence of the microscopic world: but to sceptics it was born as an unattractive theory which seemed inconsistent with existing 'facts' about the universe. There were two ways of proceeding. One could accept the counter-intuitive results of Quantum Theory as hard facts about the universe and relegate Classical Physics to an emergent approximation arising from (rather than principle foundation to) the more fundamental Quantum Theory. The alternative is to maintain a belief that Classical Physics is the more fundamental theory of nature, and that Quantum Mechanics arises as an emergent approximation to an as-yet-undescribed deeper reality. The latter position is often referred to as a 'hidden variable interpretation'.

The inception of Quantum Mechanics involved an array of ill-defined terms borrowed from the common natural language which form an essential part of describing the theory. Words like 'apparatus' are meant to refer to a common and everyday meaning of the word but actually require a tighter definition that is lacking from the axioms of Quantum Theory. John Stuart Bell was one of those not happy to accept the loose definitions presented in descriptions of Quantum Theory, famously expressing a discomfort with the term 'measurement' [2], which appears frequently both in everyday discourse and in most axiomatizations of Quantum Theory.

The majority of scientists, along with the popular literature and textbooks, quickly swallowed the new terminology, skating over the issue of the meanings of the words, their interpretation, and ultimately some of the worrying ramifications of the theory as a whole. This was likely due to the empirical success of the theory: it was powerful enough to absorb the erstwhile anomalous results of the Photoelectric Effect and the Double Slit Experiment, providing an explanation for them. Furthermore the theory makes many other predictions which enjoy a high degree of confirmation. In fact Quantum Theory would pave the way to Quantum Electrodynamics (QED), a theory which agrees with experimental data to unprecedented and unrivalled numerical precision. Not all were as easily swept up into the 'shut up and calculate'[2] camp by this sort of empirical success however, and some continued to ponder the problem of understanding the theory and reconciling it with other important theories in Physics. It is ironic and perhaps significant that QED, the best confirmed theory we have, is a Quantum Field Theory and as such is a marriage of Quantum Theory with Special Relativity: for it is Special Relativity with which Quantum Theory has the biggest disagreement. In a famous paper [10] Einstein, Podolsky and Rosen put forward an argument that Quantum Mechanics is *incomplete*, and really needed an hidden variable theory to underpin it. Whether they were correct or not is contentious, but the paper spurred on other thinkers to show explicitly that a hidden variable interpretation of Quantum Theory would have to be 'non-local' [1, 13, 12].Einstein's epithet for the notion of non-locality was "*spukhafte fernwirkung*" or "spooky action at a distance", and he had personal reasons for his uneasiness with quantum mechanical predictions. Einstein's seminal work, Special Relativity, was built around the fundamental idea of causality: he decreed that the constant speed of light provides a limit on the speed at which observers can communicate, and a finite lower bound on the time taken to transmit a causal influence. In short Einstein had stated that cause and effect could not be instantaneous (it must be local), and the Quantum Theorists had disagreed. Rather than take sides, the scientific community tends still to hold two seemingly mutually contradictory beliefs: Einstein was right, and he was wrong. This state of suspended judgement is a barrier to progress toward a unified theory of nature. Of course the jury being out does not prevent scientists from making predictions with the theory and developing technologies; but it may be preventing them from achieving much more than would otherwise be possible.

In this work I aim to present arguments that contribute to the struggle of improving the interpretation of Quantum Mechanics. The improved understanding of the logical implications of the separate conceptual and mathematical ingredients of Quantum Theory should help in the two areas discussed above: the inconsistency of the theory with other important physical theories and the indefiniteness of the language used to axiomatize it. Work on isolating conceptual ingredients of Quantum Theory may shed light on the possibility of a non-local hidden variable theory, or on interpreting the many strange results of the theory. It may also assist in improving the understanding of key words in the theory which could then be defined in a much more satisfactory way. The work might well help in strengthening intuitions about the theory: new methods of calculation might emerge and new ways of visualising the quantum world (or the hidden reality currently approximated by the quantum world).

An improvement in the clarity, consistency and detail of the axioms of Quantum Theory will help in a Philosophical regard, but importantly in an empirical regard with new predictions that may be subjected to scientific testing. New ways of describing the theory or understanding the mathematics that underpin it may well point to as yet undiscovered phenomena, or as yet undiscovered technologies which could be developed. Ultimately an insight into the nature of the theory should allow us to exploit it further, rather than merely smooth the furrowed brows of those who seek an elegant axiomatization free from contradiction with the rest of physics.

1.2 Project Outline

In section 1.3 I introduce (by way of a culinary analogy) the technique of isolating conceptual ingredients of a theory through comparison with 'toy theories'. I will discuss a particular type of toy theory: the *quantum-like theory* (a term first used in [9]), and claim that with enough quantum like theories we may be able to entirely classify and isolate conceptual ingredients and mathematical structures which give rise to quantum phenomena, and use this classification and insight to recast the theory in a way that has a clearer interpretation. In section 2 a brief recapitulation of Quantum Mechanics is given with some basic notions from the field of Quantum Information. In section 3 I will introduce Spekken's toy theory as a concrete example of a quantum like theory and case study for the suggested research programme. In section 4 Stabilizer Quantum Mechanics is introduced as a sub theory or restricted version of Quantum Mechanics which is 'closer' to Spekkens's toy theory than unrestricted Quantum Mechanics (in that it there is a bijection between the Stabilizer states and Spekkens's toy states). The existence of a no cloning theorem in both theories is discussed: the fact that an information theoretic principle is clearly responsible for the instance of the theorem in one theory suggests that it might be responsible in the other. In section 5 both Spekkens's toy theory and Stabilizer Quantum Mechanics are described using Category Theory, and Category Theoretic language and notation is used to exactly pinpoint not only where the theories are similar and where they differ but exactly how the differences in mathematical structure bring about differences in extant phenomena. I present the proof (due to Coecke, Spekkens and Edwards [8]) that if a quantum-like theory has a phase group isomorphic to $Z_2 \times Z_2$ (the Klein group) then it remains local, but if it has a phase group isomorphic to Z_4 (the four element cyclic group) then it exhibits non-locality. Finally in section 6 conclusions are drawn about the success of this case study, and future work is suggested.

1.3 Methodology

When a baker makes a cake, she combines ingredients in a bowl, mixes and heats. The resultant cake can exhibit colour, texture, shape and taste, taking on different qualities as the ingredients vary. The first cake makers were probably like the founders of Quantum Theory: stumbling across ingredients that worked in combination to produce desired results. As bakery and science have improved bakers have learnt more about the action of certain ingredients: more baking soda means a larger size, more sugar means a sweeter taste. Further it is now possible to understand at a deeper level the reason for ingredients to cause their respective qualities in the cake: the microscopic affect of egg at a molecular level can explain how this ingredient binds the cake together and impacts the texture of the cake.

Isolating Conceptual Ingredients of a Physical Theory in order to find a good set of axioms for it is like being presented with a mystery cake, and trying to retrofit an accurate recipe for it. Perhaps a vague recipe exists, but the amount

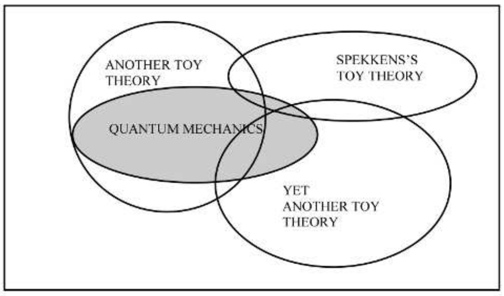


Figure 1: *Quantum Phenomena might be a subset of the union of phenomena of many toy theories. Of course any given toy theory may exhibit 'extra' or 'unwanted' phenomena which are not exhibited by Quantum Theory, and no one toy theory is likely to exactly capture all of the quantum phenomena.*

of each ingredient is undefined, or the ingredients are named ambiguously. What is a good way to proceed? The suggestion is to bake a number of 'toy cakes'. These toy cakes are so called because they are baked for fun: or more precisely they are baked to help discover something about the real mystery cake. They do not attempt to taste nice or look appealing: any desirable attributes like this are completely coincidental. A variety of toy cakes are baked with different sets of ingredients until similarities with the mystery cake are revealed. Eventually ingredients can systematically be chosen to fully reproduce the mystery cake, and now by construction an accurate recipe is known.

A toy theory is like a toy cake. It does not attempt to make statements about reality that might stand up to testing: any desirable qualities like this are purely coincidental. It is constructed just for fun, or more precisely to help highlight some facts about the real theory (in this case Quantum Theory). Conceptual Ingredients are not like egg white or sugar or flour, of course: they are mathematical structures (such as a vector space) or intellectual ideas (such as the Spin-Statistics theorem) that in combination bring about prediction of natural phenomena. It is the goal of this project to begin the process of isolating the conceptual ingredients of Quantum Theory by using various toy theories in an exactly analogous way to the cake example outlined above.

What should be sought is a complete classification of toy theories and their phenomena, where all quantum phenomena were accounted for by at least one toy theory (see Figure 1).

2 Quantum Mechanics

2.1 A Set of Axioms

The Copenhagen Interpretation of Quantum Mechanics has axioms [14]:

Axiom 1. *At a fixed time the state of a quantum system is a vector $|\psi\rangle$ in a Hilbert Space $\mathcal{H}$.*

Axiom 2. *The evolution of a quantum system is always described by unitary evolution, i.e. by a Unitary ($\hat{U}^\dagger = \hat{U}^{-1}$) operator or 'gate' $\hat{U}$ with $|\psi\rangle \to \hat{U}|\psi\rangle$. In particular the time evolution obeys the Schrödinger Equation*[1]

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{1}$$

with $\hat{H}$ a unitary operator known as the Hamiltonian.

Axiom 3. The Born Rule. *A measurement on a quantum system is described by an Hermitian operator ($\hat{A} = \hat{A}^\dagger$) known as an* Observable. *An observable has spectral decomposition*

$$\hat{A} = \sum_i \lambda_i |e_i\rangle\langle e_i|. \tag{2}$$

The eigenvalues λ_i of an Observable are the possible values obtainable by measurement[2]*, and the $|e_i\rangle\langle e_i|$ are known as* projectors. *When a state $|\psi\rangle$ is measured the probability of getting outcome i is*

$$p(i) = \langle\psi||e_i\rangle\langle e_i|\psi\rangle. \tag{3}$$

Axiom 4. The Collapse Postulate. *The post measurement state of the system is*

$$\frac{|e_i\rangle\langle e_i|\psi\rangle}{\sqrt{p(i)}}. \tag{4}$$

Axiom 5. Completeness Relation *The projectors satisfy*

$$\sum_i |e_i\rangle\langle e_i| = \mathbb{I}. \tag{5}$$

where $\mathbb{I}$ is the identity operator.

2.2 Quantum Information

The primitive object of Quantum Information theory is a two level system described by a two dimensional Hilbert Space. A large edifice of protocols, algorithms and architectures can be built on this primitive object as a foundation, in

[1] I set Planck's constant $\hbar = 1$

[2] In this project the eigenvalues are always assumed to always be non-degenerate

the same way that the modern digital computer and all its complexities is built upon the foundation of a bit. A bit is the smallest measure of information. It can distinguish between two possibilities. It can be realised physically in many ways: any classical binary (two level) system will do, such as the result of an ideal coin flip, the on/off status of a bulb, or the high/low current status of a wire in an integrated circuit. One of the possibilities is labelled as 'logical zero'

$$0 \tag{6}$$

and the other as 'logical one'

$$1. \tag{7}$$

If we wish to represent more information then we can consider a 'string' of many bits, and the number of possibilities now rises as a power of two. For example if we consider 3 bits then we can distinguish between the $2^3 = 8$ possibilities associated with the strings $000, 001, 010, 100, 011, 101, 110, 111$.

A 'Quantum - Bit' or *qubit* is the quantum mechanical equivalent of a classical bit. It is the simplest non trivial Quantum Mechanical system. Upon measurement it distinguishes between two possibilities. It can be realised physically in many ways: the first/second energy level of the Hydrogen atom, the spin up/spin down state of a spin half particle, the horizontal/vertical polarization of a photon. *The difference is that between measurements the qubit exists as a superposition of the two possibilities.* We cannot say which value is taken on by the system between measurements. The two possibilities that are obtainable by measurement are in correspondence with elements of a basis of the two dimensional Hilbert Space. The canonical choice for notational reasons is the computational basis, with e.g. spin up (one of the possibilities) as

$$|0\rangle \tag{8}$$

and spin down (the other possibility) as

$$|1\rangle. \tag{9}$$

Another important difference between bits and qubits is that one can make measurements on a qubit in a direction other than one that distinguishes $|0\rangle$ from $|1\rangle$. A general qubit exists in a state known as a 'coherent superposition':

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{10}$$

with α, β complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ if the state is normalised $\langle\psi|\psi\rangle = 1$. Importantly states which differ by a global phase are identified, since they produce identical physical predictions. This enables a specification of a qubit state by a relative weight and a relative phase. Considering for now only equal weight superpositions of two qubits we have

$$|\psi\rangle = \sqrt{2}^{-1}(|0\rangle + e^{i\theta}|1\rangle), \tag{11}$$

the following states being special cases with $\theta = 0, \pi$ respectively:

$$|+\rangle = \sqrt{2}^{-1}(|0\rangle + |1\rangle)$$
$$|-\rangle = \sqrt{2}^{-1}(|0\rangle - |1\rangle).$$

It is possible to make a measurement distinguishing $|+\rangle$ and $|-\rangle$ by defining the observable

$$\hat{X} = |+\rangle\langle+| - |-\rangle\langle-|. \tag{12}$$

The outcome of a measurement will not always be wholly determined by the measurement itself and the state $|\psi\rangle$. This is one of the keystone ideas of quantum mechanics which sets it apart from any classical physics.

The information contained in a classical binary state is quantified by a bit, and this is the amount of information required to specify the state completely. But how might a quantum state be specified? A little thought soon uncovers an interesting fact: the complex coefficients α and β can take on a continuum of non-integer values. *There is an infinite amount of classical information involved in specifying a quantum state.* This is just one paradox [4] which drives a wedge between classical theories and quantum mechanics. An altogether novel concept of 'Quantum Information' arises.

Definition The density matrix ρ_ψ associated with a qubit state $|\psi\rangle$ is a 2×2 matrix satisfying

$$\rho_\psi = |\psi\rangle\langle\psi| \tag{13}$$

or equivalently if $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ then

$$\rho_\psi = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \beta^*\alpha & |\beta|^2 \end{pmatrix}. \tag{14}$$

Theorem 2.1. *The normalisation condition for a state implies that it's density matrix representation has unit trace.*

Definition The *fidelity* between two quantum states $|\psi\rangle$ and $|\chi\rangle$ is defined by $\mathcal{F} = |\langle\psi|\chi\rangle|^2$ or for two states with density matrices ρ and σ the fidelity is $\mathcal{F} = \mathrm{Tr}\sqrt{\rho}\sqrt{\sigma}$.

Definition Two quantum states $|\psi\rangle$ and $|\chi\rangle$ are said to be *orthogonal* iff $\langle\psi|\chi\rangle = 0$. This is a special case of Fidelity when $\mathcal{F} = 0$.

Definition A state that can be written as a linear combination of basis vectors is called a pure state.

Definition The *coherent superposition* of two or more pure states is another pure state:

$$|\psi\rangle = \sum_i \alpha_i |i\rangle \tag{15}$$

with the $\alpha_i \in \mathbb{C}$ known as *weights*.

Definition The *convex combination* ρ of two or more quantum states is a mixed state defined as the linear combination of density matrices representing those states with probabilities p_i that sum to unity:

$$\rho = \sum_i p_i \rho_i \tag{16}$$

and there are many convex decompositions of a mixed state: for example the maximally mixed state $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|) = \mathbb{I}/2$. A mixed state cannot be written as a sum of basis states with complex weights.

The semantic difference between a 'pure' state and a 'mixed' state is due to the fact that the uncertainty for the former is purely quantum uncertainty, whereas for the latter some classical uncertainty is introduced. A mixed state would be produced if a quantum state is prepared based on the result of a classical coin flip, with different states for heads and tails. Alternatively the classical uncertainty can be understood as some sort of experimental error.

There is an elegant representation of qubits which greatly enhances a quantum physicist's visualisation of measurement, evolution, and the closeness of states in Hilbert Space. In the density matrix representation of a pure state there are ostensibly two complex, or four real degrees of freedom. The normalisation condition (that density matrices have unit trace) removes one real degree of freedom leaving three real degrees of freedom. As we have seen pure states which differ only by a global phase are identified with each other, and this removes a further degree of freedom. This means there are only two remaining degrees of freedom for a pure state, or three remaining degrees of freedom for a mixed state: so mixed states can be represented in a volume and pure states on a surface.

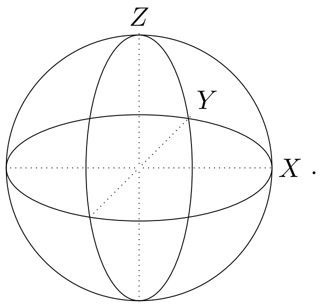
Definition The Bloch Vector $\vec{r} = [r_x, r_y, r_z]$ associated with any quantum state ρ satisfies

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{r}.\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} \tag{17}$$

with $\vec{\sigma} = [\hat{X}, \hat{Y}, \hat{Z}]$ a vector of Pauli matrices. The Bloch vector enables us to plot any qubit state in a three dimensional space known as the Bloch Sphere. Pure states lie on the surface of the sphere, and mixed states in it's interior. Orthogonal states are antipodal points. By construction eigenstates of the Pauli Matrices form three sets of antipodal points

$$\hat{X}|+\rangle = \ \ |+\rangle \tag{18}$$
$$\hat{X}|-\rangle = -|-\rangle \tag{19}$$
$$\hat{Y}|i^+\rangle = \ \ |i^+\rangle \tag{20}$$
$$\hat{Y}|i^-\rangle = -|i^-\rangle \tag{21}$$
$$\hat{Z}|0\rangle = \ \ |0\rangle \tag{22}$$
$$\hat{Z}|1\rangle = -|1\rangle \tag{23}$$

with $|i^{\pm}\rangle = \sqrt{2}^{-1}(|0\rangle \pm i|1\rangle)$. Each pair of eigenstates is a basis for the Hilbert space and defines an orthogonal axis in the Bloch sphere:



(24)

Note that orthogonal here does not mean that the bases are orthogonal: they are mutually unbiased. This means that the inner product of the elements of any two bases is a constant: the geometrical consequence is that mutually unbiased states are equidistant in the Bloch sphere.

The geometrical representation is an incredibly useful one: it enables a visual comparison of many effects within Quantum Information Theory and also the states (and their evolution) of any quantum-like theory can be compared readily in a visual manner. Convex combination maps pure states from the surface of the Bloch sphere to its mixed states in it's interior, or mixed states in it's interior to other mixed states in its interior. The maximally mixed state, defined as $\frac{\mathbb{I}}{2}$ lies at the centre of the Bloch sphere.

2.3 Pairs of Qubits

When there is a need to describe multiple systems, this is achieved by using the tensor product $\otimes$. For example if we have a Qubit in New York that we want to describe mathematically and at the same time a Qubit in London, we can write:

$$|\psi\rangle_{\text{New York}} \otimes |\phi\rangle_{\text{London}}, \tag{25}$$

where the qubit in New York is in the state labelled by $|\psi\rangle$, the Qubit in London is in the state labelled by ϕ, and $\otimes$ is the tensor product. It is common practice, once an order convention has been established, to drop the subscript labels from the qubits, and even conflate the two kets to form a larger one:

$$|\psi\rangle_{\text{New York}} \otimes |\phi\rangle_{\text{London}} = |\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle. \tag{26}$$

The expression (25) is referred to as a 'product state', and it exists in the tensor product Hilbert space. That is to say, if $|\psi\rangle \in \mathcal{H}_1$ with basis $\{|N\rangle, |Y\rangle\}$ and $|\phi\rangle \in \mathcal{H}_2$ with basis $\{|L\rangle, |D\rangle\}$ then $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ with basis $\{|NL\rangle, |ND\rangle, |YL\rangle, |YD\rangle\}$.

It is possible to pass the product state through a gate, just as it is with a single qubit. This gate can itself be the tensor product of two separate gates acting on the separate qubits respectively:

$$(\hat{A}_{\text{New York}} \otimes \hat{B}_{\text{London}})|\psi\rangle \otimes |\phi\rangle. \tag{27}$$

If we allow $\hat{A}$ to be the identity operator then (27) describes acting only on the London Qubit, and likewise if we allow $\hat{B}$ to be the identity operator then it describes acting only on the New York Qubit.

The usual computational basis for qubits is extended to a composite system of a pair of qubits (existing in a *bipartite* Hilbert space): the basis elements are denoted $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A general bipartite state is thus a linear combination of these basis elements. An extremely important caveat here is that bipartite quantum states may evolve into special 'entangled' states. For example there is the Bell state

$$|\text{Bell}\rangle = \sqrt{2}^{-1}(|00\rangle + |11\rangle). \tag{28}$$

Notice how in the computational basis expansion of the Bell state certain 'cross-terms' $|01\rangle$ and $|10\rangle$ are missing (they have zero weight), and that consequently this state may not be written as the product of two single qubit states. Entangled states such as this are central to many Quantum Information Processing protocols.

2.4 No-Cloning Theorem

Now that a bipartite product space has been introduced, it can be used to show-off some of the strange and surprising phenomena of quantum theory. Consider the copying or cloning of a quantum state. A universal state cloner acts on both a data qubit $|\psi\rangle$ and an ancilla qubit in some arbitrary fixed state, producing a final bipartite state that is a product of two copies of the data qubit. For argument's sake suppose the ancilla qubit is in a 'blank' state of $|0\rangle$. A cloner achieves

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle \tag{29}$$

and by the same process (for a universal cloner)

$$U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle \tag{30}$$

for a distinct state $|\phi\rangle$. Taking the inner product of (29) and (30) yields

$$\langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2 \tag{31}$$

which has solutions only if the states are identical (inner product is one) or orthogonal (inner product is zero). Hence a universal state cloner is impossible. The only inner product preserving cloning process is for a single state or for two orthogonal states. The hidden argument is that the cloning process is not unitary (it does not preserve inner products) and as such is not allowed by Axiom 2.

In order to clone a state, the cloner or the cloning mechanism must be able to access all of the information specifying the state without disturbing it. Classical states can be specified by a finite amount of information and completely survive the process of measurement; this means that they may be cloned simply by identifying the state through measurement and preparing a fresh system in an identical state. Perhaps the reason that quantum states cannot be cloned in this way is because an infinite number of measurements are required to completely identify a quantum state. It is more likely that the reason is that quantum states are destroyed by measurement, and hence may never be identified from an arbitrary ensemble of states.

2.5 Teleportation

Teleportation is another surprising example of a quantum phenomenon. It is a result that meshes with the no-cloning theorem: one cannot clone a state without *simultaneously destroying it*.

The description of the protocol which enables teleportation necessarily requires a tripartite Hilbert Space. This is easily understood as the tensor product of three Hilbert spaces.

Let's imagine that Alice holds a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that she wishes to send to Bob, and that there is no quantum channel in which to do so (i.e. she is unable to send the physical instantiation of the qubit). Let us imagine further that the identity of the state (the amplitudes α and β) are unknown to both parties so that classical communication of the identity of the state is impossible. There is however a classical communication channel open to Alice and Bob (perhaps a telephone), and local measurements and operations are allowed. In order to implement the protocol Alice and Bob share a Bell state:

$$|\text{teleport}\rangle = |\psi\rangle|\text{Bell}\rangle = \sqrt{2}^{-1}|\psi\rangle(|00\rangle + |11\rangle). \tag{32}$$

This state exists in the tripartite Hilbert space (because the Bell state is a bipartite state). The notational convention will follow:

$$\mathcal{H} = \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_2}_{\text{Alice}} \otimes \underbrace{\mathcal{H}_3}_{\text{Bob}}, \tag{33}$$

i.e. Alice holds both the state to be teleported and one 'half' of the Bell state with Bob holding the other half. We can rewrite (32) as

$$|\text{teleport}\rangle = \sqrt{2}^{-1}(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle) \tag{34}$$

$$= \sqrt{2}^{-1}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle). \tag{35}$$

Alice now proceeds by making a measurement corresponding to the observable $A = \sum_{i=\mathbb{I},X,Y,Z} \lambda_i P_i$ which has four possible outcomes of equal likelihood associated with the following projectors:

$$P_{\mathbb{I}} = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \tag{36}$$
$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \tag{37}$$
$$P_X = \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|) \tag{38}$$
$$= \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|) \tag{39}$$
$$P_Y = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) \tag{40}$$
$$= \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) \tag{41}$$
$$P_Z = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) \tag{42}$$
$$= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|). \tag{43}$$

It is a simple exercise to check the following facts: that $P_I + P_X + P_Y + P_Z = \mathbb{I}$ in accordance with Axiom 5, that each of the outcomes has probability of a quarter in accordance with the Born Rule (Axiom 3), and that the post measurement states are

$$|\text{teleport}'_{\mathbb{I}}\rangle = P_{\mathbb{I}} \otimes \mathbb{I}|\text{teleport}\rangle = |\text{Bell}\rangle(|\psi\rangle) \tag{44}$$
$$|\text{teleport}'_{\sigma}\rangle = P_{\sigma} \otimes \mathbb{I}|\text{teleport}\rangle = |\text{Bell}'\rangle(\sigma|\psi\rangle) \tag{45}$$

Where $|\text{Bell}'\rangle$ is the Bell state or a unitary rotation of the Bell state and $\sigma = \hat{X}, \hat{Y}, \hat{Z}$ is a Pauli operator. Notice that in a quarter of cases, teleportation has been achieved. Alice now telephones Bob and transmits two bits of classical information, informing him of the result of her measurement λ_i: for the outcome associated with $P_{\mathbb{I}}$ she transmits a coded message with meaning "teleportation achieved" and for the outcome associated with P_σ she transmits one of three coded messages identifying an appropriate Pauli Operator. Bob then uses this information to rotate his qubit to the state desired by applying the relevant Pauli Operator:

$$\mathbb{I} \otimes \mathbb{I} \otimes \sigma|\text{teleport}'_{\sigma}\rangle = |\text{Bell}\rangle|\psi\rangle, \tag{46}$$

where the involution of the Pauli operators is assumed $\sigma^2 = \mathbb{I}$. This completes the description of a protocol which, through the paradigm of Local Operation and Classical Communication has consistently achieved:

$$|\psi\rangle|\text{Bell}\rangle \rightarrow |\text{Bell}'\rangle|\psi\rangle. \tag{47}$$

This protocol will be revisited in Section 5.5 where it is described in an altogether different language.

3 Spekkens's Toy Theory

3.1 Introduction

This section is chiefly a summary of a toy theory presented by Spekkens in [15]. Spekkens's is a classical theory equipped with an information theoretic principle: there is restriction on the amount of knowledge one can obtain about any given system. This restriction is dubbed *The Knowledge Balance Principle* and it has surprising consequences. Spekkens's toy theory may, some believe, bridge the gap between an intuitive and straightforward explanation of reality and the strangeness of Quantum Theory. As the toy theory is explicitly a local theory, the Bell theorem [1] implies that it cannot reproduce all of Quantum Mechanics. By construction, then, the toy theory must have a role other than a theory of nature. It does not benefit from any sort of experimental confirmation, and so it is deserving of toy theory status. It does qualify as a quantum like theory - it has a no cloning theorem and many other phenomena that are considered characteristic of Quantum Mechanics. Spekkens would have us take this as evidence for an *epistemic view* of quantum states; an interpretative point that differs from the mainstream. The epistemic view of quantum states views the statements of Quantum Theory not as the ultimate description of reality, but as a statement of incomplete knowledge about some *hidden* reality. Spekkens's view that there may be a contextual or non-local hidden variable interpretation of Quantum Mechanics [16] is largely irrelevant to the research programme of this project. The toy theories, of which Spekkens's is an example, could help us understand and refine orthodox views irrespective of their own success.

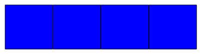
3.2 The Elementary System

The simplest system of the toy theory is called an elementary system. An elementary system can be in one of four states, and these are fact-of-the-matter states of reality called *ontic states*. They are the distinct values taken on by a classical discrete degree of freedom[3]. The ontic state is an abstract entity, and the arguments of this project (and indeed of arguments for the epistemic view of quantum states) go through irrespective of a concrete definition. It can aid understanding however to think of a tetrahedral die as one example of a physical system with four classical states.

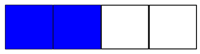
An elementary system is represented by a row of four boxes, each representing a distinct ontic state (such as each face of the tetrahedral die). An opinion about the system, known as an *epistemic* state, is represented by a disjunction of ontic states: for instance $a \vee b$ (read 'a or b') expresses the knowledge that a system is either in ontic state a or ontic state b. Equivalently we can shade in boxes corresponding to ontic states that the system could be in, and express

[3]The true classical notion of a state is a point in phase space. In this discussion we would have a phase space that allows only four possibilities for the values of the canonical variables: perhaps there is a classical particle with fixed momentum and four possible positions. See section 3.11.

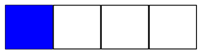
our knowledge that way. In this way we can represent total ignorance:

$$1 \vee 2 \vee 3 \vee 4 \tag{48}$$


partial knowledge,

$$1 \vee 2 \tag{49}$$


or complete knowledge

$$1 \tag{50}$$


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Definition The Ontic base of an epistemic state is the set of ontic states that are compatible with it. For example the ontic base of $(1 \vee 2 \vee 3 \vee 4)$ is $\{1, 2, 3, 4\}$.

Theorem 3.1. *For an elementary system, if there are n ontic states in the ontic base of an epistemic state, then the probability distribution is uniform over these ontic states and each square in the graphical representation represents a probability of $\frac{1}{n}$.*

In (48) each shaded square represents a probability of $\frac{1}{4}$, in (49) each shaded square represents a probability of $\frac{1}{2}$ and in (50) the single shaded square has unit probability. When we have complete knowledge our epistemic state coincides with the ontic state of the system. In the same way that when a probability distribution on the possible values of a variable is a Delta function this corresponds to certainty about the value of that variable.

3.3 The Knowledge Balance Principle

It is interesting to pause here and ask: which features if any of Quantum Mechanics could be exhibited by the entirely classical system thus far outlined? The answer is obviously that only 'classical phenomena' would be reproduced. Quantum Theory is not fully classical, but it is undoubtedly built upon classical foundations, and has classical phenomena. 'Classical phenomena' is a rather clumsy phrase, because it describes something completely uninteresting. Quantum Mechanics may exhibit non-classical phenomena such as non-commuting observables, but it also falls back on classical notions frequently. When measurements are made on a particle, in fact, the quantum world disappears for an instant and the particle momentarily has a classically defined position and momentum. The axioms of Quantum Theory supplement and restrict classical notions, enabling the theory to arrive at its bizarre conclusions. A toy theory can place restrictions on classical notions and lead to conclusions in precisely the same way. The same is true of Spekkens's toy theory. That the conclusions generated resemble those of Quantum Theory suggests that the conceptual ingredient(s) that restricts or extends classical notions is culpable for the interesting phenomena shared by the theories. The conceptual ingredient that concerns us here is the *Knowledge Balance Principle*.

The Knowledge Balance Principle roughly states that we can never achieve more than half of the complete amount of knowledge. To make this principle precise we must give a quantitative description of knowledge.

Definition A *Canonical Set* is a set containing the minimal number of yes/no questions needed to fully specify the ontic state of a given system.

An inefficient way of specifying a quantum state for an elementary system in ontic state 1,2,3 or 4 would be for example

$$\left\{ \begin{array}{c} \text{`Is it 1?'}, \\ \text{`Is it 2?'}, \\ \text{`Is it 3?'}, \\ \text{`Is it 4?'} \end{array} \right\}, \tag{51}$$

whilst the more efficient canonical set might be

$$\left\{ \begin{array}{c} \text{`Is it 1 or 2?'}, \\ \text{`Is it 2 or 3?'} \end{array} \right\}. \tag{52}$$

Clearly this example of a canonical set is capable of precisely ascertaining the ontic state, with the minimal number of yes/no questions. This is because each question halves the possibilities for the identity of the ontic state. There are other canonical sets than this one for the same system, but what is important is the number of questions in the canonical set (which is constant for a given system). Here, for a single elementary system of four ontic states, the number is two.

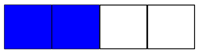
Definition *Spekkens's Measure of Knowledge* is the maximum number of questions for which the answer is known in a variation over all canonical sets of questions (i.e. we pick the canonical set which contains the most number of answered questions, and this number constitutes the amount of knowledge about a system).

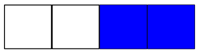
Definition *Spekken's Measure of Ignorance* is the difference between the total number of questions in a canonical set and the amount of knowledge.

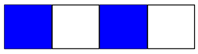
Theorem 3.2. The Knowledge Balance Principle: *in a state of maximal knowledge, for every system and at every time the amount of knowledge one possesses about the ontic state of a system at that time must equal the amount of ignorance about that system* [15].

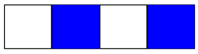
The immediate consequence of Theorem 3.2 is that maximal knowledge is incomplete. It is like having nature roll a tetrahedral die, and allow us only incomplete knowledge of the identity of the label on the uppermost vertex.

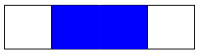
For a single elementary system we have two questions in a canonical set, one of which must remain unanswered. Recalling the form of the questions in the canonical set (52), the principle means we may discount at most two ontic states at any given time. This corresponds to a minimum of two squares remaining shaded in the pictorial representation of epistemic states. Hence an exhaustive list of the six epistemic states of maximal knowledge can be given for a single elementary system:

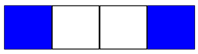
$1 \vee 2$  (53)

$3 \vee 4$  (54)

$1 \vee 3$  (55)

$2 \vee 4$  (56)

$2 \vee 3$  (57)

$1 \vee 4$  (58)

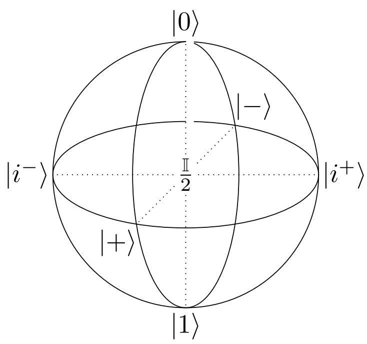
with a state of maximal ignorance (48) completing the set of seven allowed epistemic states for a single elementary system. States of maximal knowledge will be known as pure toy bits. An epistemic state of maximal knowledge is known as a 'toy bit' as it takes a single classical bit to answer one question in a canonical set for an elementary system.

3.4 Single Qubit Pure State Analogues

It is possible to set up a dictionary between the seven states of the toy system and seven important states from Quantum Information theory; the ± 1 eigenstates of the Pauli matrices and the maximally mixed state:

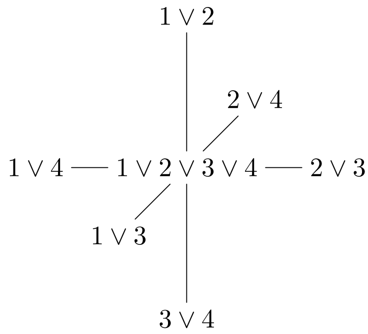
$$\begin{aligned} 1 \vee 2 &\Leftrightarrow |0\rangle \\ 3 \vee 4 &\Leftrightarrow |1\rangle \\ 1 \vee 3 &\Leftrightarrow |+\rangle \\ 2 \vee 4 &\Leftrightarrow |-\rangle \\ 2 \vee 3 &\Leftrightarrow |i^+\rangle \\ 1 \vee 4 &\Leftrightarrow |i^-\rangle \\ 1 \vee 2 \vee 3 \vee 4 &\Leftrightarrow \mathbb{I}/2. \end{aligned} \tag{59}$$

Considering the Bloch Sphere representation of qubits:



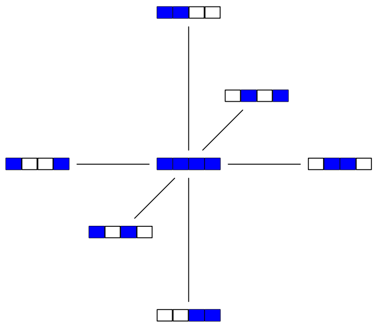
(60)

using the dictionary (59) we can construct a geometrical representation of states in the toy theory



(61)

or alternatively



(62)

which we will refer to as the *toy bit insect.*

Definition Two states are *disjoint* if the intersection of their ontic bases is null.

Using the dictionary (59) or the geometrical representations (61) and (62) one can appreciate that states that are analogous to *orthogonal* states in Quantum Theory are *disjoint* in the toy theory. Three pairs of antipodal points on the Bloch Sphere become three pairs of points 'opposite' each other in the toy bit insect. If one superimposes two disjoint states on each other one notices no overlap between shaded boxes.

Definition The *fidelity* $\mathcal{F}[\bar{p},\bar{q}]$ between two epistemic states $(a \vee b \vee c \vee d)$ and $(e \vee f \vee g \vee h)$ with p_k a vector of uniform probabilities for a,b,c,d which sum to unity and q_k a vector of uniform probabilities for e,f,g,h which sum to unity is :

$$\mathcal{F}[\bar{p},\bar{q}] = \sum_k \sqrt{p_k}\sqrt{q_k}. \tag{63}$$

Fidelity is a measure of nondisjointness. Disjointness and Equality are special cases of fidelity when $\mathcal{F} = 0, 1$ respectively.

If one takes the Quantum Fidelity between any two states on the right hand side, and the Toy Theory Fidelity between the corresponding states on the left hand side of the dictionary (59) one finds a numerical agreement for each pairing.

Definition The *convex combination* of two disjoint epistemic states is the union of their ontic bases if the union forms a valid epistemic state. Otherwise, and for nondisjoint states, the convex combination is undefined.

Definition The coherent superpositions of two disjoint epistemic states $(a \vee b)$ and $(c \vee d)$ with $a, b \neq c, d$ and $a < b, c < d$ are defined:

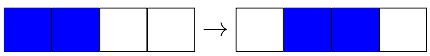
$$\begin{array}{rcl|l} (a \vee b) +_1 (c \vee d) & = & (a \vee c) & \text{low-low} \\ (a \vee b) +_2 (c \vee d) & = & (b \vee d) & \text{high-high} \\ (a \vee b) +_3 (c \vee d) & = & (b \vee c) & \text{high-low} \\ (a \vee b) +_4 (c \vee d) & = & (a \vee d) & \text{low-high} \end{array} \tag{64}$$

with the mnemonic in the right hand column codifying how to construct each coherent superposition.

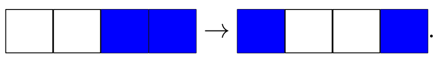
Coherent binary operations map pure toy bits to other pure toy bits (generating an analogue to the coherent superpositions of Quantum Mechanics), and so in the toy bit insect representation we can think of this as mapping between any two of the sharp ends of the six protruding legs to any of the other four sharp ends. The four coherent binary operations $+_1, +_2, +_3, +_4$ are analogous to equal weight superpositions of two pure states in Quantum Theory (11) with a relative phase θ chosen from $\{0, \pi, \pi/2, 3\pi/2\}$ respectively.

3.5 Transformations

In the toy theory, any transformation of the states must be such that it preserves fidelity. Fidelity preserving operations are exactly analogous to unitary transformations in Quantum Theory, which preserve the inner product. The allowed transformations must be permutations of the ontic states. This is necessary to prohibit transformations which map many ontic states to a single ontic state, which is a map from a legal epistemic state to an illegal one which violates the Knowledge Balance Principle. For example the transformation



can be achieved by a (123)(4) permutation (that is a cyclic permutation of the ontic states 1, 2, 3 and no permutation of ontic state 4). The same permutation achieves



Note that the fidelity between initial states is the same as the fidelity between final states. This is true for all of permutations of the ontic states, which from the 24 element group S_4.

3.6 Measurement

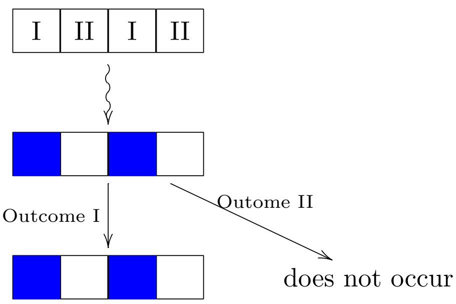
In Quantum Theory a measurement involves both an outcome and a post measurement state (which is the eigenvector of the observable with the measurement outcome as its eigenvalue). For the post measurement state in the toy theory to obey the Knowledge Balance Principle it must rule out all ontic states incompatible with the measurement outcome. A measurement will act on an elementary system of four ontic states and can distinguish between two sets of two ontic states, with the outcome determining the post measurement state. There are $\binom{4}{2} = 6$ partitioning of four states into two sets of two, but half of them are merely a relabelling and as the order is irrelevant this leaves three possibilities for a measurement on an elementary system. We denote a measurement of "$a \vee b$ versus $c \vee d$" by using roman numerals as labels:

$$\begin{array}{|c|c|c|c|}\hline \mathrm{I} & \mathrm{I} & \mathrm{II} & \mathrm{II} \\ \hline \end{array} = \{1 \vee 2, 3 \vee 4\} \tag{65}$$

$$\begin{array}{|c|c|c|c|}\hline \mathrm{I} & \mathrm{II} & \mathrm{I} & \mathrm{II} \\ \hline \end{array} = \{1 \vee 3, 2 \vee 4\} \tag{66}$$

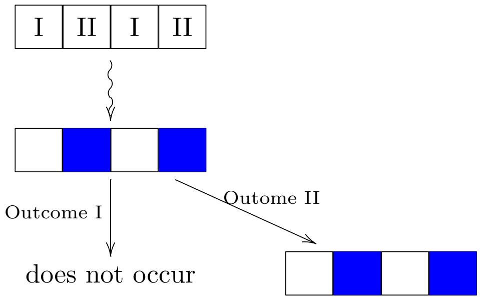
$$\begin{array}{|c|c|c|c|}\hline \mathrm{I} & \mathrm{II} & \mathrm{II} & \mathrm{I} \\ \hline \end{array} = \{1 \vee 4, 2 \vee 3\}. \tag{67}$$

The Roman numerals label the measurement "direction" in analogy to projective measurements in Quantum Theory. Disjoint epistemic states are distinguishable by a suitable choice of measurement direction, just as orthogonal states are distinguishable in Quantum Theory. A measurement direction defined by "$1 \vee 3$ versus $2 \vee 4$" can distinguish between epistemic states $1 \vee 3$ and $2 \vee 4$ with certainty. Diagrammatically I represent a measurement by a wiggly arrow pointing from the measurement direction to the system being measured. This visual way of representing measurements may help build intuitions about the measurement process. The post measurement states corresponding to the different outcomes of the measurement are given:



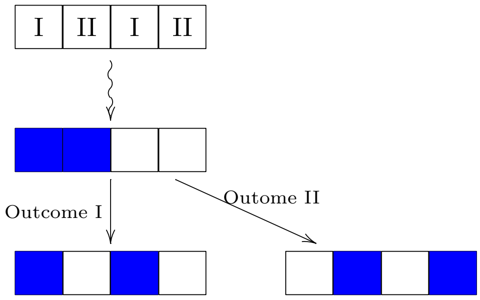
. (68)

The post-measurement state is always given by an epistemic state coinciding with the appropriate sector of the measurement direction, provided there is a non zero fidelity between this zone and the pre-measurement state. If the Fidelity is zero then that outcome is impossible. To complete this example I show the other epistemic state which, with the same measurement direction has a certain outcome, allowing the measurement to distinguish between these epistemic states:



(69)

If it is known that the epistemic state is either $1 \vee 3$ or its disjoint $2 \vee 4$ then the above measurement can distinguish between these states, by virtue of the determination of the measurement outcome by the epistemic state. Some measurement outcomes are not determined by the initial epistemic state and the measurement direction :



(70)

Only the relative frequencies of outcomes in a large ensemble of experiments is determined. The important point is that although we have some uncertainty here in analogy with the Quantum Theory, it can be rested squarely on the shoulders of a lack of knowledge! There is no need in the toy theory to swallow the bitter pill of accepting uncertainty as a fundamental aspect of reality, but merely as the run-of-the-mill classical uncertainty as incomplete knowledge. It is possible to tell a story explaining a deterministic transformation of the ontic states of the system whilst maintaining a lack of knowledge about the identity of those ontic states and a lack of knowledge about their transformation. Consider the 'I Outcome' in (70): it seems as if the intersection of pre and post measurement states would allow one to infer with certainty the ontic state as 1 and violate the Knowledge Balance Principle. Importantly, the Knowledge Balance Principle says nothing of the possibility of retrodicting a past ontic state and hence there is no violation unless we can know the identity of the current instantaneous ontic state. To maintain a balance of Knowledge we must conclude that the ontic state has been transformed by the measurement from 1 to *either* 1 or 3. Our uncertainty about which transformation has taken place secures a valid post epistemic state, whilst allowing for a deterministic and well defined value for the ontic state at all times, albeit hidden from our perceptions.

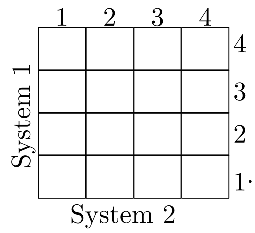
3.7 Pairs of Elementary Systems

Theorem 3.3. *Every system is built from elementary systems. For n systems there are $2n$ questions in the canonical set and 2^{2n} possible ontic states, because each question in a canonical set halves the possibilities for the ontic state.*

If we consider two elementary systems simultaneously, the ontic states are every possible pairing of each of the ontic states on the separate systems in analogy with the tensor product. There are sixteen possibilities, and the joint states are written $(a \cdot b)$ with a representing the ontic state of the first system and b the second system:

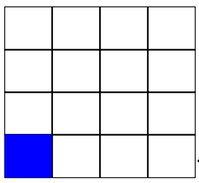
$$\left\{ \begin{array}{llll} (4\cdot 1), & (4\cdot 2), & (4\cdot 3), & (4\cdot 4), \\ (3\cdot 1), & (3\cdot 2), & (3\cdot 3), & (3\cdot 4), \\ (2\cdot 1), & (2\cdot 2), & (2\cdot 3), & (2\cdot 4), \\ (1\cdot 1), & (1\cdot 2), & (1\cdot 3), & (1\cdot 4) \end{array} \right\}. \tag{71}$$

We can represent the sixteen ontic states on a grid, with the first system's ontic state being a row and the second system's ontic state being a column:



(72)

Note how the labelling of boxes mimics neither the usual labelling of coordinates in the Cartesian plane (because the horizontal axis is the second label not the first) nor of matrix elements (labels are read up and then right, rather than row and then column). As an example consider the joint ontic state of "system 1 in ontic state 1 and system 2 in ontic state 1" written as (1.1) or depicted



(73)

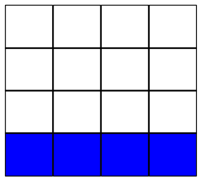
This is not a valid candidate for an epistemic state for two reasons. Firstly the 'marginal' epistemic states do not obey the Knowledge Balance Principle; that is to say that if we look at at one of the systems in isolation, ignoring information about the other state the amount of knowledge expressed in (73) would allow more than half of the questions in a canonical set for that subsystem to be answered. This way of ignoring some of the information in a bipartite state is in analogy with the partial trace of Quantum Mechanics (see e.g. [14]), and in the toy theory the marginal distributions are found by projecting all of the rows into a single sub-row (if we discard information about System 1) or projecting all of the columns into a single sub-column (if we discard information about System 2). Secondly if we consider the Knowledge Balance Principle as it applies to the composite system, and also consider Theorem 3.3 which implies for $n = 2$ that we may know the answers to a maximum of two out of four questions in the canonical set, states such as (73) violate the principle.

Theorem 3.4. *There must be a minimum of four boxes which are shaded on a composite system of two elementary subsystems.*

This means there are

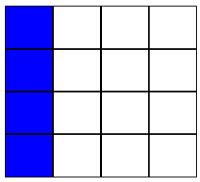
$$\binom{16}{4} = 1820 \tag{74}$$

states of maximal knowledge satisfying the Knowledge Balance Principle on the the joint system. Some of these states will violate the principle as it applies to a subsystem, however. Consider for example



(75)

which will violate the Knowledge Balance System on the second subsystem and



(76)

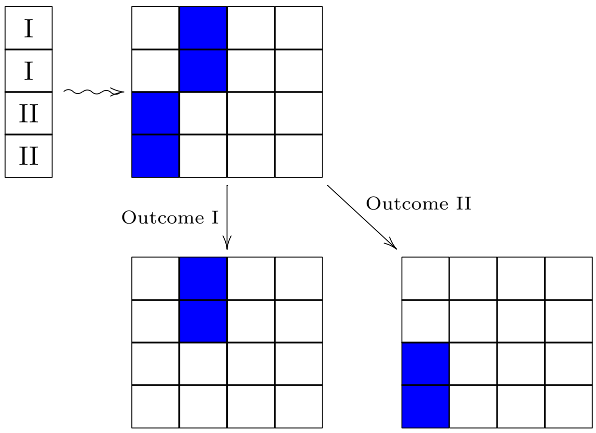
which will violate the Knowledge Balance Principle on the first subsystem. If we look at the marginal distributions, they are invalid epistemic states. There are other invalid states which are invalid because they will be updated in some cases to a post-measurement state which violates the Knowledge Balance Principle (as it applies to the composite system) for particular measurements.

3.8 Measurements on Bipartite States

Definition A *Product State* in the toy theory is one in which a maximal amount of knowledge about each subsystem is possessed, given the Knowledge Balance Principle and it's application to composite and both subsystems. If System 1 has marginal $(a \vee b)$ and System 2 has marginal $(c \vee d)$ then the product state is simply $(a \vee b) \cdot (c \vee d) = (a \cdot c) \vee (b \cdot c) \vee (a \cdot d) \vee (b \cdot d)$, where the disjunction has been distributed over the conjunction[4].

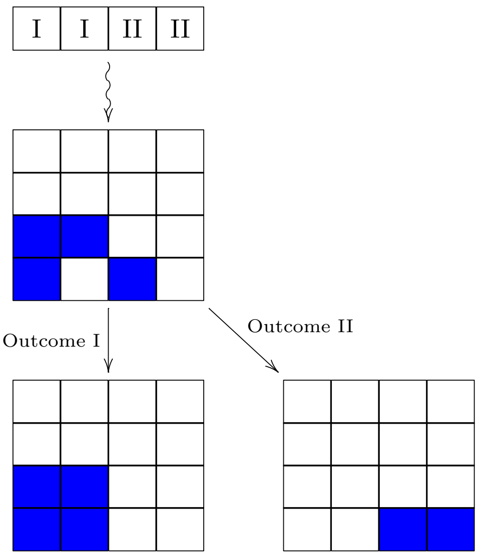
[4]In much the same way as we distribute addition over multiplication in $(x + a)(y + b) = xy + ay + xb + ab$.

The post measurement state for two elementary systems is calculated in the usual way, except now it is only the marginal distribution which is updated on whichever subsystem undergoes measurement. The other subsystem's marginal is updated to the state consistent with the measurement outcome and the pre-measurement state (i.e. the intersection of the ontic bases of the pre-measurement state and the appropriate measurement direction zone). The joint state is simply a product state of the two updated marginals:



. (77)

Similarly



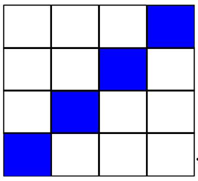
. (78)

Note that in (78) the post measurement state depends on the measurement outcome, and in one case the post measurement state is valid and obeys the Knowledge Balance Principle. Crucially however, there is a possibility that the other outcome is found and the Knowledge Balance Principle is violated. Such a possibility is excluded by removing these (pre-measurement) states from the theory.

3.9 Correlated States

Definition A *Correlated State* in the toy theory is one in which complete knowledge about the relation between each subsystem is possessed, but nothing about the ontic states of the separate subsystems is known. Both systems will have a marginal that represents total ignorance, and the bipartite state will be $(1 \cdot 1) \vee (2 \cdot 2) \vee (3 \cdot 3) \vee (4 \cdot 4)$ or state found by permutations of the rows or columns of this state.

An example of a correlated state:



. (79)

It should be clear that a correlated state may not be written as the product of two marginal distributions. This is precisely because it must lack certain "cross terms" in analogy with the entangled states of Quantum Mechanics (see Section 2.3).

3.10 Cardinality of States

For a composite system of n elementary systems, there are 4^n ontic states and 2^n questions in a canonical set. If we answer the maximum of half of these 2^{n-1} this corresponds to 2^n ontic states in an epistemic state of maximal knowledge. Therefore the number of epistemic states satisfying the Knowledge Balance Principle on the composite system is

$$\bar{\Omega}_n = \binom{4^n}{2^n}. \tag{80}$$

For two elementary systems ($n = 2$) this amounts to 1820 states, but as I have pointed out most of these will be invalid.

Theorem 3.5. *The only valid bipartite states of maximal knowledge (pure bipartite toy bits) in the toy theory are product states and correlated states.*

Using Theorem 3.5 we can calculate how many pure toy bit states there are for two elementary systems. As a product state can be constructed by picking two rows from four and then two columns from four columns and shading the intersection there are

$$\binom{4}{2} \cdot \binom{4}{2} = 36 \tag{81}$$

product states. For the correlated states one must shade in a single box in any of the four rows of the first column, then shade in a box in one of the other three rows of the second column, and then shade in a box in one of the two remaining rows in the third column. The final box to be shaded must be the one in the final column in the final remaining row. Hence there are

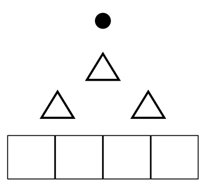
$$4! = 24 \tag{82}$$

correlated states. This makes a total of 60 pure toy bit states for two elementary systems.

3.11 A Physical basis for the principle?

In this section I present a suggestion for a physical system with a single classical degree of freedom (in this case position) which reproduces the abstract concept of an elementary system and hints as to a possible physical reason for the Knowledge Balance Principle. The aim is not to argue for a particular microscopic account of the hidden reality, but to explore a possibility and consider its plausibility. The exercise will highlight key questions about the toy theory: is a physical basis for the information theoretic principle necessary or can such a principle be basic or primitive, underlying any physical principles? The exercise also serves as a proof that it is possible for the ontic state to be specified in a totally classical and easy to understand way, and for uncertainty to melt away as a true aspect of reality (it being relegated to the status of a lack of knowledge).

In Section 3.2 we saw that the elementary system could be represented physically by a tetrahedral die. This works insofar as there are four possibilities for the ontic state of the system (the lowermost face or uppermost vertex of the tetrahedral die). It is difficult (though not impossible) to form a physical basis for the knowledge balance principle for this system, and to imagine how nature might prevent us from gaining complete knowledge about the die. Instead I opt for a row of four boxes with three wedges positioned above it, which makes for a very simple interfacing with the pictures already in use to represent states:


. (83)

There is a classical particle or 'ball' which begins above the top wedge. The ball will fall under the affect of gravity and be imparted a lateral force by two of the three wedges. The ball can fall into any of the four boxes by interacting with two of the wedges. The final position of the ball dictates the ontic state. Each wedge has a bias: either to the left or to the right, and the lower wedges always have the same bias. The Knowledge Balance Principle for this system would imply that the possible amounts of knowledge attainable would be knowledge of the bias at the higher level *or* knowledge of the bias at the lower level *or* neither *but not* both. Knowledge of the bias at both the upper level and lower level would lead to complete knowledge of the final position of the ball, the ontic state. It should be clear that all of the epistemic states for a single elementary system introduced above can be assigned to this physical system if one bit of information[5] is given about the bias of the wedges (either at the upper or lower level). The state of total ignorance of course can also apply to this physical system. This suggestion for the physical basis of the toy theory sharpens questions about the physical basis of the no cloning theorem in the theory, if there is such a physical basis.

3.12 No-cloning Theorem

In the toy theory, cloning or copying a state of incomplete knowledge is essentially the ability to apply such an epistemic state to a new system whilst maintaining its applicability to the original system. It is "not about duplicating parts of reality" [16], because epistemic states are not part of the landscape of reality but rather parts of the landscape of our language or of our mind. In section 2 we saw that classical states of reality may be cloned, but quantum mechanical states may not be. We shall now see that toy states may not be cloned.

A toy bit cloner would perform

$$(a \vee b).(c \vee d) \rightarrow (a \vee b).(a \vee b) \tag{84}$$

where $(a \vee b)$ is the data toy bit and $(c \vee d)$ is an arbitrary ancilla toy bit. The same transformation would, for a universal cloning process, have to achieve

$$(e \vee f).(c \vee d) \rightarrow (e \vee f).(e \vee f) \tag{85}$$

for any other data toy bit $(e \vee f)$. Taking the fidelity between (84) and (85) gives:

$$\mathcal{F}[(a \vee b), (e \vee f)] \rightarrow \mathcal{F}[(a \vee b), (e \vee f)]^2 \tag{86}$$

where I have assumed

$$\mathcal{F}[(a \vee b).(c \vee d), (e \vee f).(c \vee d)] = \mathcal{F}[(a \vee b), (e \vee f)] \times \mathcal{F}[(c \vee d), (c \vee d)] \tag{87}$$

in analogy with the Hilbert space inner product. Notice that (86) implies that a fidelity preserving cloning process exists only for either a single state or for disjoint states. As the only transformations allowed in the toy theory are those preserving fidelity (see Section 3.5), a universal cloning process is therefore impossible.

The mathematical similarity with the no-cloning theorem for quantum states is manifest. Cloning is described and then shown to be an illegal transformation. Toy bits by definition are specified by a small finite amount of information: so

[5] It is straightforward to cast this example in terms of canonical sets and questions answered as in previous discussions.

heuristically the 'infinitude of information' argument above for quantum states cannot be given for toy bits: coherent quantum states not only correspond to an uncountably infinite amount if classical information, but they are also destroyed by measurement[6]. It seems that it is the conceptual ingredient of 'information gain implies disturbance' that is instrumental. A possible solution would be to determine the identity of the toy bit to be cloned, and then prepare two 'fresh' systems in this state. But determining an arbitrary state in the toy theory requires more than one measurement for certainty and since measurements disturb the toy bits, a second or subsequent measurement is useless and states cannot be identified with confidence (see Section 3.6).

[6] An argument could be made that both classical and quantum states can be defined approximately by a finite amount of information by 'gridding' the phase space or Hilbert space of the given system [4], thus fixing the problem of infinite information. The problem of measurement disturbing the state remains.

4 Stabilizer Formalism

Despite the similarities shared by the toy bit theory with Quantum Mechanics, there are some important differences. Perhaps the most important is that the toy theory is discrete where Quantum Mechanics has a continuum of states, measurements and transformations. Happily there is a version of Quantum Mechanics which also has discrete pure states and transformations and as such promises to share a stronger similarity with Spekkens's toy theory.

Stabilizer Quantum Mechanics is a restricted version of Quantum Mechanics. It admits allows only a subset of states and transformations of Quantum Mechanics.

Definition *An operator M* stabilizes *a state $|\psi\rangle$ iff $M|\psi\rangle = |\psi\rangle$. Equivalently if a state is the $+1$ eigenstate of some operator then the state is said to be stabilized by that operator.*

Definition *The Pauli Group on n-qubits G_n is the group of n-fold tensor products of Pauli matrices and/or Pauli matrices scaled by ± 1, $\pm i$.*

For the single qubit case:

$$G_1 \equiv \{\pm\mathbb{I}, \pm X, \pm Y, \pm Z, \pm i\mathbb{I}, \pm iX, \pm iY, \pm iZ\}\,. \tag{88}$$

G_n is the product group of n lots of G_1:

$$G_n = \underbrace{G_1 \times \ldots \times G_1}_{n \text{ of these}}\,. \tag{89}$$

A subgroup of G_n defines a subspace V_s which is the spanned by the set of all states simultaneously stabilized by all elements of the subgroup:

$$V_s := \{|\phi\rangle : S_i|\phi\rangle = |\phi\rangle \quad \forall S_i \in S\}\,. \tag{90}$$

An example of a subgroup of G_n is

$$S = \{\mathbb{I}, X_1X_2, X_2X_3, X_1X_3\} \tag{91}$$

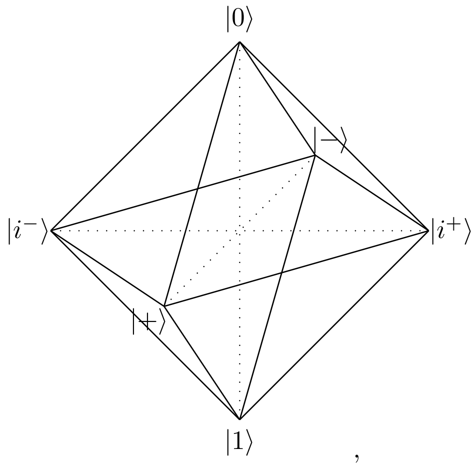
with the notation $\mathbb{I} = \mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}$ and $X_1X_3 = X\otimes\mathbb{I}\otimes X$ etc. The subspace stabilized by this subgroup is

$$V_s = \{|\chi\rangle \,|\, |\chi\rangle = \alpha|+++\rangle + \beta|---\rangle\} \qquad \alpha,\beta\in\mathbb{C} \tag{92}$$

and this subspace is the intersection of the subspaces stabilized by each of the elements of the subgroup S. Stabilizer Quantum Mechanics has only states that exist in a subspace stabilized by subgroups of the Pauli Group, and transformations that map these states into each other. These transformations are the *Clifford Unitary gates.*

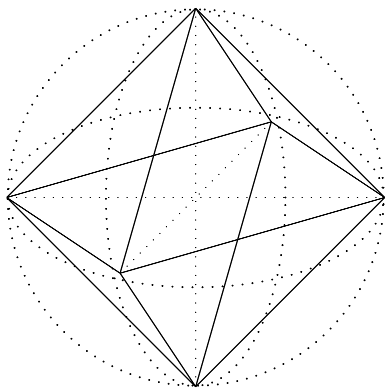
4.1 Stabilizer Octahedron

There is a useful geometrical representation of Stabilizer states for one qubit::



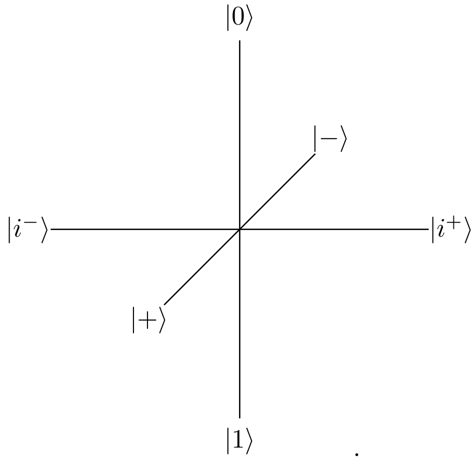
, (93)

upon which the Bloch ball can be superimposed



. (94)

The places that the ball touches the octahedron identify the pure stabilizer states[7]:



. (95)

Notice the similarity with the toy bit insect (62). As the states inherit all of their properties from Quantum Mechanics, there is now a manifest bijective correspondence between pure stabilizer states and toy bits for a single elementary system or qubit.

4.2 No Cloning Theorem

The theorem is inherited from unrestricted Quantum Mechanics, since the stabilizer states are a subset of quantum states and the Clifford transformations are a subset of unitary transformations.

4.3 Cardinality of states

The number of distinct stabilizer states for N-qubits is [3]:

$$\Omega_N = 2^{N(N+3)/2} \prod_{k=1}^{N} (1 + 2^{-k}) \tag{96}$$

so for one qubit there are $\Omega_1 = 6$ distinct stabilizer states, and are the six $+1$ eigenstates of $\pm X, \pm Y, \pm Z$ to mirror the states on the right hand side of the dictionary (59). For two qubits there are

$$\Omega_2 = 60 \tag{97}$$

states, which fall into 24 entangled states and 36 product states in the same way as Spekkens's toy bits. Immediately we have a stronger similarity with the

[7] Strictly speaking 'mixed stabilizer state' is a contradiction in terms, as such a state is not in fact stabilized by members of the Pauli group. We can imagine, however, a mixed state in unrestricted Quantum Mechanics which has a convex decomposition in terms of stabilizer states.

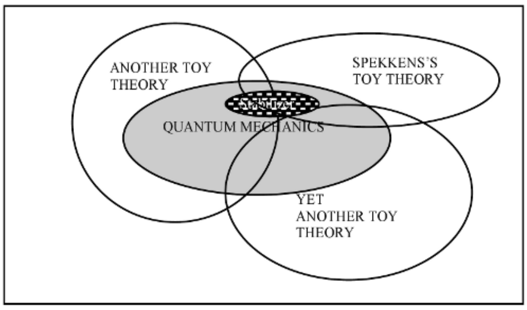


Figure 2: *Stabilizer Quantum Mechanics exhibits phenomena which are a subset of the phenomena of unrestricted Quantum Mechanics. The analogy with Spekkens's toy theory is stronger than was the case with unrestricted Quantum Mechanics, but there are still some effects which are unaccounted for.*

toy theory than was enjoyed with unrestricted quantum mechanics. See Figure 2.

4.4 GHZ Argument for Non-Locality

There is an elegant argument against the possibility of a hidden variables interpretation of Quantum Mechanics, and it is presented succinctly in [13]. Stabilizer Quantum Mechanics inherits this argument. The key ingredient is the tripartite generalisation of the Bell state, known as the GHZ state:

$$|\mathrm{GHZ}\rangle = \sqrt{2}^{-1}(|000\rangle + |111\rangle). \tag{98}$$

Note that this is a stabilizer state. This state is a simultaneous eigenstate of the observables $X \otimes X \otimes X, X \otimes Y \otimes Y, Y \otimes X \otimes Y$ and $Y \otimes Y \otimes X$ which are all tripartite tensor products of individual Pauli operators and furthermore are admitted to the stabilizer formalism by virtue of being Clifford Unitary gates. The observables have eigenvalues of ± 1. A hidden variable interpretation will attempt to assign a value of either $+1$ or -1 to each individual system for each Pauli operator. The interpretation will then be that the system possesses these values as an aspect of reality and that a measurement merely reveals the value of these pre-existing variables as eigenvalues of the observables. For example

$$XYY|\mathrm{GHZ}\rangle = -1|\mathrm{GHZ}\rangle \tag{99}$$

and the realist is likely to claim that either the triple of separate hidden variables corresponding to XYY are either

$$\begin{array}{ccccl} (+ & - & +) & \text{or} \\ (- & - & -) & \text{or} \\ (- & + & +) & \text{or} \\ (+ & + & -) & \end{array} \tag{100}$$

where + stands for +1 and − stands for −1 and is the *parity* of the eigenvalue of the observable. Notice this list exhausts all possible assignments of hidden variables to XYY, since each individual observable can only return ±1 on measurement, and the parities must multiply to give −1. We can repeat this procedure for XXX, YXY and YYX, and in each case there will be four possible assignments of hidden variables. Considering two observables each with with two eigenvalues on the three qubits, there are $2^{2^3} = 64$ ways of simultaneously assigning hidden variables to all four observables (remembering that a particular Pauli operator should be assigned the same hidden variable if it acts on the same qubit). We could check each possibility for consistency, but there is a simpler way. If we list the observables in a 'Mermin Table' [9]:

Observable			*Eigenvalue*
X	X	X	+
X	Y	Y	−
Y	X	Y	−
Y	Y	X	−
+	+	+	?

(101)

we notice that in each column we have two observables appearing twice. If the individual observables are to have pre-existing values of ±1 then these must be the same for the same qubit (the same column). Hence the product of the parities in each column are necessarily +1 (the column parities are all the product of two squares of ±1). The table parity can be calculated by multiplying the column parities, giving +1.

Each observable has an eigenvalue for the GHZ state which can be checked by direct computation. The eigenvalues are listed as row parities, and the table parity can be found by multiplying these row parities, giving −1. This is a contradiction, and we are left with only two resolutions. Either accept that no hidden variable assignment is possible for the GHZ state and Observables listed, or claim that the value of an individual observable on one qubit can be instantaneously affected by a measurement on a different qubit which may be separated by a space like interval from the first. It is this affect that is known as *Non-locality.* The argument has succeeded in concluding that there can be no local hidden variable interpretation of Quantum Mechanics precisely because such an interpretation cannot be applied without contradiction for the GHZ state and observables above.

5 Category Theory

Using category theory, Bill Edwards suggests a programme of classification for quantum like theories that might elucidate which features and important results of Quantum Theory are consequent from which axioms or assumptions [9]. As a proof of this programme, he examines Spekkens's toy theory: in particular its relation to stabilizer quantum mechanics. In fact Spekkens's original paper [15] hints at this sort of a research programme:

> "A distinction between those quantum phenomena that are due to maximal information being incomplete and those quantum phenomena that arise from some other conceptual ingredient is likely to be very useful in the field of quantum information theory, where there is currently a paucity of intuitions regarding what sorts of information processing tasks can be implemented more successfully in a quantum universe than in a classical universe". [15]

In our case the quantum phenomenon under consideration is non-locality, and the "other conceptual ingredient" turns out to be something called the *phase group*. The facet of non-locality is explored in a paper examining Stabilizer Quantum Mechanics and Spekken's toy theory authored by Spekkens, Coecke and Edwards [8]. They state (emphasis in original):

> "It is our goal... to identify the *piece of structure* of Hilbert space quantum mechanics that *generates non locality.* To this end we will use the framework [of category theory] to analyse two theories which make very similar predictions, but differ principally in that one is local and the other is non-local."

We will see that some phase groups generate non-locality while others do not.

5.1 What is Category Theory?

Category theory is a way of thinking in a unified way about disparate areas of mathematics. The idea is that it generalizes over things like vector spaces, Hilbert spaces, sets, groups and other mathematical structures to form the *more abstract idea* of an **Object**. It also generalizes over things like functions, group homomorphisms, linear maps and other mathematical processes that are associated with the Objects to form the more abstract idea of a **Morphism**. The two ideas of Object and Morphism considered together are a **Category**. The claim from the Category Theorists is that their 'way of thinking' is sufficiently general to be able to describe (at least) both quantum mechanics and Spekken's toy theory using the same precise language and that this generality will make more obvious the similarities and differences in structure of the two theories. The Objects of a category are really just labels for physical systems. The Morphisms can link two Objects with a direction, for example turning eggs into an omelette[8]. We can express that a morphism or an arrow f links objects A (the domain) and B (the co-domain) in a number of different ways. Firstly there is a notation which mirrors the notation for sets and maps:

$$f : A \to B. \tag{102}$$

Alternatively we may express that f is a member of the set of all morphisms between A and B (known as the Hom-Set) in the Category $\mathcal{C}$

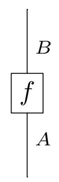
$$f \in \mathcal{C}(A, B). \tag{103}$$

Another way of denoting expressions is diagrammatically, with objects as capital letters, and morphisms as arrows with a lower case letter:

$$A \xrightarrow{f} B\ , \tag{104}$$

alternatively we can use a straight line for an object and a box for a morphism, and use an input/output type picture:

(105)



In all cases note that the order of the Objects is important, i.e.

$$f : A \to B \quad \neq \quad f : B \to A \tag{106}$$
$$f \in \mathcal{C}(A, B) \quad \neq \quad f \in \mathcal{C}(B, A) \tag{107}$$

Similarly we must distinguish between morphisms that have the same domain and co-domain: boiling an egg is a distinct process to frying it, although the end product is a cooked egg in both cases:

$$f : A \to B \neq g : A \to B \tag{108}$$

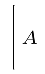
The notations (102),(103),(104),(105) are all completely equivalent, but lend themselves more naturally to different situations. I will concentrate on the input/output notation for the rest of the project. These diagrams are read 'upwards' from bottom to top.

For each Object there is a special Morphism known as the **Unit Morphism,** which does nothing to the object:

$$A \xrightarrow{1_A} A \tag{109}$$

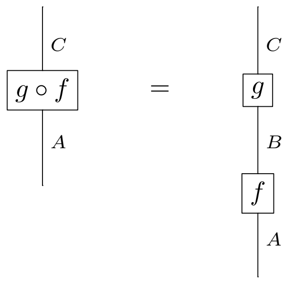
[8]The fact is that one must break some eggs in order to make an omelette, but this step is ignored.

or in the input / output notation



(110)

We can express *successive application of morphisms* algebraically $g \circ f$ (read 'f then g') or diagrammatically:



. (111)

This is like frying the egg *and then* seasoning the egg to produce an seasoned omelette. Whilst the diagrams are read upwards, the expressions in the algebraic representation are read 'backwards' or right to left (as is standard in e.g. linear algebra). The equality of the diagrams above means that any two successive morphisms (between say Objects A and B , B and C respectively) can be viewed as a single morphism (between A and C). We also have *associativity* $(g \circ f) \circ h = g \circ (f \circ h)$.

5.2 †SMCs

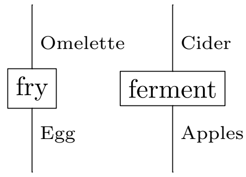
Along with the concepts of Object and Morphism that make up a Category, it is necessary to introduce further concepts or structures to construct a 'Dagger Symmetric Monoidal Catgeory' or †SMC. This entity is a Category with some extra conceptual apparatus.

Definition A Functor is a map on Objects *and* a map on Morphisms. A Bifunctor is a functor taking two arguments.

There is a Bifunctor $(\_ \otimes \_)$ called the 'tensor' which is a way of considering Objects and Morphisms in parallel (as opposed to the serial nature of successive application above). So we can fry the egg and ferment the apples independently but in parallel:

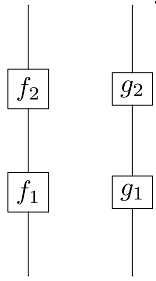
$$\text{Egg} \otimes \text{Apples} \xrightarrow{\text{fry} \otimes \text{ferment}} \text{Omelette} \otimes \text{Cider} \tag{112}$$

or



. (113)

The **Bifunctoriality** condition reads $(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$ and the condition is implicit in the pictorial calculus . *Both* sides of this equality are given by the same diagram:



. (114)

This is the statement that '(frying the egg and then seasoning the egg) whilst (fermenting the apples then sweetening the apples)' is equivalent to '(frying the egg whilst fermenting the apples) then (seasoning the egg whilst sweetening the apples)'.

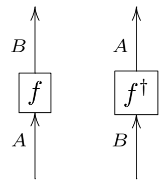
If the definitions and concepts introduced here seem familiar, this should not be surprising: Category Theory claims to be a generalisation of Hilbert space quantum mechanics, and as such has a similarity with it. The aspects of Category Theory that are more surprising are still to be introduced.

There is an **Identity Object** which, in the input/output notation is depicted as a blank space. It represents the absence of a system. This is not to say it represents a physical vacuum but rather a lack of a physical *state*: and viewing states as information of some kind we can think of the Identity Object as being an unspecified state.

There is a **Dagger Structure** which means lines, or wires (which are Objects) are endowed with a direction which we can reverse by taking the dual $A \rightarrow A^*$:

(115)

Also each morphism $f : A \to B$ has a dagger morphism or adjoint morphism $f^{\dagger} : B \to A$ which can be considered as an 'upside down box':

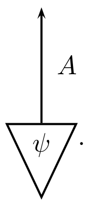


(116)

The dagger is a *contravariant* functor which is also 'identity on objects'. This means it turns outputs into inputs and inputs into outputs, but has no effect on the objects themselves.

5.3 Categorical Quantum Mechanics

Quantum Mechanics is a concrete physical theory and hence is associated with a category. Pure state Quantum Mechanics arises as the Category **FHilb.** In this category Objects are finite Hilbert Spaces representing physical systems (i.e. a qubit or a pair of qubits or a qutrit etc.), and Morphisms are linear maps (i.e. Unitary Transformations or Evolutions). The tensor is the usual tensor product and the dagger is the adjoint operation. As special cases of morphisms we have States or 'State Preparations' which transform an unspecified state into a specified state: i.e. they are a map from the Identity Object into another Object of the Category. They have at least one output but no input:

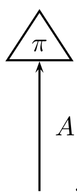


. (117)

Recall that we read diagrams from bottom to top (logical concatenation or succession in time). As the name suggests they are very much in analogy with the states or kets of quantum mechanics

$$|\psi\rangle. \tag{118}$$

Similarly we have the case where a morphism transforms the state of an existing system into an unspecified state: i.e. a map from an Object of the Category to the Identity Object. These are 'co-states' and have input but no output:

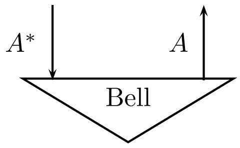


. (119)

And these are like the dual states or bras of quantum mechanics:

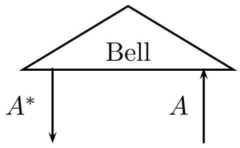
$$\langle \pi |. \tag{120}$$

An obvious question here might be: "why are states depicted as morphisms rather than objects?" Whilst it is true that kets seem to have more to do with entities which undergo processes than the processes themselves, we should remind ourselves that Objects are the Hilbert Spaces themselves rather than elements of the Hilbert Space (vectors or kets). Hence what we have represented is the process of preparing system A in state ψ in the first case, and in the second case destroying a state through for example making a measurement. To demonstrate this further we introduce the 'Bell state' or 'entanglement generator':



(121)

and the bell costate



(122)

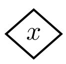
as a state preparation with no input and *two* outputs and a state deletion with *two* inputs and no outputs respectively. Recall that the directions of the arrows do not denote time flow or label whether an object is the input or output. For now I will vaguely suggest that the arrows denote the flow of quantum information. These pictures are of course analogous to

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{123}$$

and

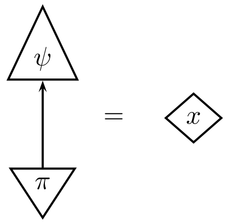
$$\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) \tag{124}$$

from quantum mechanics (here we work in the computational basis but any basis $\{|a_i\rangle\}$ gives rise to a bell state $\sum_i |a_i\rangle \otimes |a_i\rangle$). Finally we have *numbers* or *scalars* which have no input or output at all:



. (125)

These are the result of combining a state with a co state, or preparing a state then measuring it:



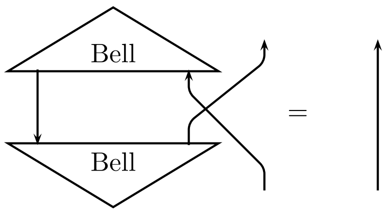
. (126)

Again the pictorial calculus aims to be easy to intuit: one can imagine even a young child pushing two triangles together to form a diamond [5]. There is even a similarity with Dirac notation: in Quantum Theory we have the Born rule playing this role of joining states with their duals, and the scalars are complex numbers:

$$\langle \pi | \psi \rangle \in \mathbb{C} \tag{127}$$

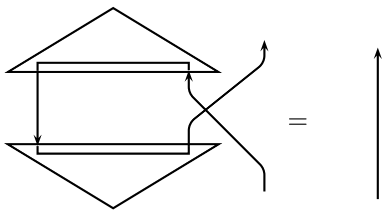
5.4 Map State Duality

There is a single axiom for the calculus:



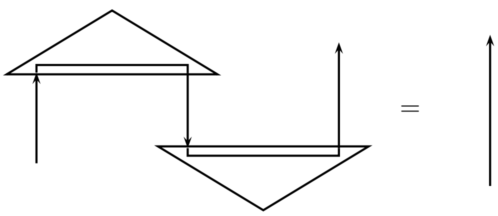
. (128)

If we imagine a rope looped through the triangles then this axiom is equivalent to 'yanking' the rope [5] to straighten out the loop:



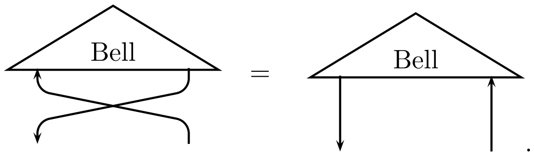
. (129)

Now we can manipulate the ends of the rope to obtain an equivalent identity:



, (130)

where we have implicitly assumed:



(131)

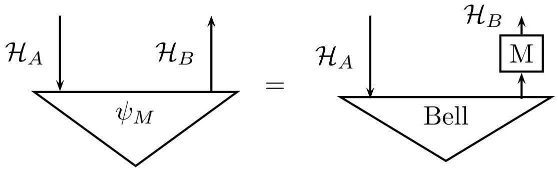
The motivation for the axiom can be seen in Dirac Notation [9]. For each state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ in a bipartite tensor product space we can expand it thus

$$|\psi_M\rangle = \sum_{ij} M_{ij} |a_i\rangle \otimes |b_j\rangle \tag{132}$$

with $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ bases for the first and second Hilbert spaces respectively. The same matrix elements M_{ij} can be used to specify a linear map taking states from the first Hilbert space to the second $M : \mathcal{H}_A \to \mathcal{H}_B$

$$M = \sum_{ij} M_{ij} |b_j\rangle\langle a_i|. \tag{133}$$

This is known as "map-state duality". We may now draw



(134)

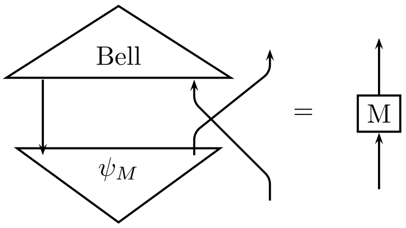
which expresses that a general bipartite state can be created by the action of applying an identity operator on one of the Hilbert spaces of a Bell state and the M operator on the other space. In Dirac notation

$$(\mathbb{I}_A \otimes M_B) \sum_i |a_i\rangle \otimes |a_i\rangle = \sum_{ij} M_{ij} |a_i\rangle \otimes |b_j\rangle \tag{135}$$

Now by applying the first 'half' of a Bell co-state to the left of (after) the first half of the bipartite state, and applying the second half of the Bell co-state to the right of (before) the second half of the bipartite state we recover M:

$$\left(\sum_k \langle a_k| \otimes \mathbb{I}\right) \left(\sum_{ij} M_{ij} |a_i\rangle \otimes |b_j\rangle\right) \left(\sum_p \mathbb{I} \otimes \langle a_p|\right) = \sum_{kj} M_{kj} |b_j\rangle\langle a_k| \tag{136}$$

pictorially we have

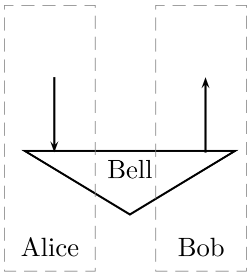


(137)

which has (128) as the special case with $M = \mathbb{I}$.

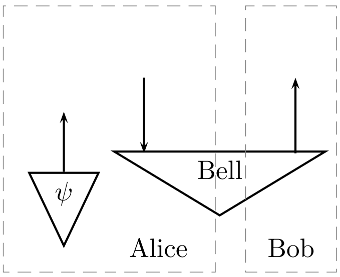
5.5 Teleportation

We can use (130) to describe teleportation [5] in a novel way. The standard version in Dirac Notation is given above in the Section 2.5. Consider Alice on the left and Bob on the right, sharing a Bell state:



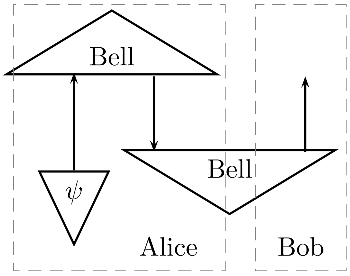
. (138)

Alice now is presented with a state to be cloned, ψ:



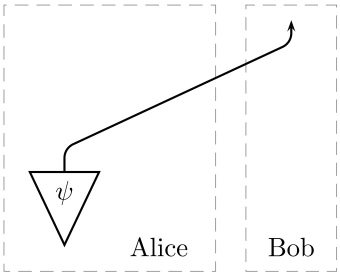
, (139)

and performs a joint measurement on the state to be cloned and her half of the bell state, using a Bell co-state:



. (140)

The axiom of the pictorial calculus (130) means we can infer teleportation[9]:



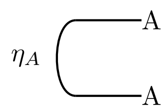
(141)

The arrow represents the flow of quantum information and this picture shows the information flowing from Alice to Bob. So there we have it: an example of the pictorial calculus in action. The claim is that the pictorial calculus is very natural in describing quantum phenomena: indeed teleportation is described succinctly above when compared with the traditional description in Dirac notation. I will go on to describe some more conceptual apparatus and further tools in the pictorial calculus that enable us to go deeper into the nature of non-locality.

[9]In actuality this describes only one of the four possibilities for a Bell co-state type measurement by Alice. For the full description of teleportation, see [6].

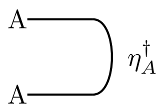
5.6 Basis Structures

We still haven't finished learning about the additional conceptual apparatus needed to complete the pictorial calculus. One step we can take to simplify things is to restrict attention to objects that are self adjoint $A = A^\dagger$. This means we no longer have to worry about the directions of any of the wires in the diagram, although we still have the ability to take the dagger of a morphism and reverse its direction. We will also orient the diagrams so they are now read left-to-right in the natural way for western script instead of reading them upwards[10]. Next we will use, instead of the Bell state (121), a morphism η called the 'unit'



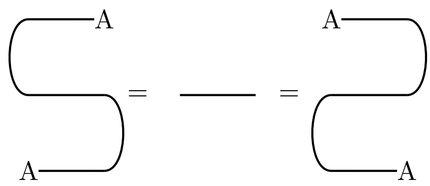
(142)

and instead of the Bell costate (122), a morphism $\eta^\dagger$ called the 'counit'



(143)

To make perfectly clear this change in the pictorial calculus, consider the axiom (130) in the new convention [9]:



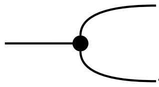
(144)

We also represent states as circles from now on, instead of triangles. Now we introduce the *Basis Structure* of an object A, which is a triple:

$$\Delta = \{A, \delta : A \to A \otimes A, \epsilon : A \to I\} \tag{145}$$

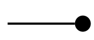
[10]Both conventions are abundant in the literature.

In other words it is the collection of A together with two special morphisms. The first special morphism $\delta : A \to A \otimes A$ can be thought of as a copying procedure and is depicted



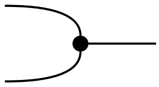
. (146)

The second morphism $\epsilon : A \to I$ as a deletion procedure, depicted



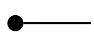
. (147)

Of course we will also have $\delta^\dagger$



, (148)

and $\epsilon^\dagger$



(149)

to consider. In **FHilb** Objects are Hilbert Spaces and morphisms are linear operators.

Definition A Hilbert space $\mathcal{H}$ with basis $\{|i\rangle\}_{i=1\ldots N}$ will have a *Basis Structure* $\Delta = \{\mathcal{H}, \delta, \epsilon\}$ where

$$\begin{aligned} \delta : \mathcal{H} &\to \mathcal{H} \otimes \mathcal{H} && (150)\\ |i\rangle &\mapsto |i\rangle \otimes |i\rangle && (151)\\ \epsilon : \mathcal{H} &\to \mathbb{C} && (152)\\ |i\rangle &\mapsto 1. && (153) \end{aligned}$$

It should be clear that there will be exactly one distinct basis structure for each distinct orthonormal basis for $\mathcal{H}$, i.e. basis structures are in bijective correspondence with orthonormal bases. As such they will often be referred to as 'observables' because they are the category theory abstract counterpart to the orthogonal projectors which make up Observables in Quantum Theory. We should expect then, to be able to see counterparts of the three mutually unbiased (orthogonal) observables X, Y, Z in categorical quantum mechanics.

The X, Y, Z observables each have eigenstates $x_\pm, y_\pm, z_\pm$ respectively: this means the Categorical counterpart, the basis structure, will only 'copy' these eigenstates. Of course for unrestricted quantum mechanics there are an infinitude of other bases that could form basis structures. For the remainder of the project we shall concentrate on a basis structure that copies the $\{|0\rangle, |1\rangle\}$ basis. The copying procedure will only successfully copy certain special states, which are the eigenstates of the basis structure, and these are the elements of the orthonormal basis that a basis structure is in correspondence with:

$$\delta :: |0\rangle \mapsto |00\rangle \tag{154}$$

$$:: |1\rangle \mapsto |11\rangle \tag{155}$$

$$:: |+\rangle \mapsto (|00\rangle + |11\rangle) \neq |++\rangle. \tag{156}$$

Notice that states that are not copied by δ are mapped to entangled states.

The fact that there is a copying procedure in this pictorial calculus strikes at the heart of the difference between Hilbert Space Quantum Mechanics in Dirac Notation, and Categorical Quantum Mechanics. Category theory enables us to keep track of information flow to and from unspecified systems in much the same way as statistical mechanics allows the flow of energy to and from a heat bath. In Categorical Quantum Mechanics we can imagine a collection systems of interest embedded in an ensemble of many such systems. The copying procedure does not imply the existence of a cloning procedure for arbitrary quantum states (in fact this would be in serious disagreement with quantum theory). The pictures used in Categorical Quantum Mechanics are very similar to the circuit diagrams abundant in literature on Quantum Information [14]. They are read left to right, and multipartite states are split across horizontal sectors of the diagram.

5.7 Basis Structure Monoid

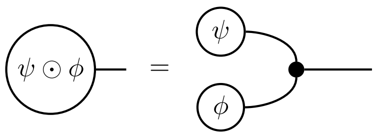
Next, we must introduce the basis structure multiplication map $(\_ \odot \_)$:

$$(\_ \odot \_) : \mathcal{C}(I, A) \times \mathcal{C}(I, A) \to \mathcal{C}(I, A) \tag{157}$$

where

$$\psi \odot \phi = \delta^\dagger \circ (\psi \otimes \phi) \tag{158}$$

or in the latest edition of the pictorial calculus:

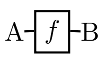


(159)

Definition The *upper star* $(\_^*)$ on a state f is defined as $f^* := (1_A \otimes \eta_B) \circ (1_A \otimes f \otimes 1_B) \circ (\eta_A^\dagger \otimes 1_B)$.

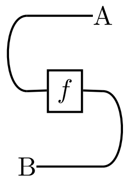
Definition The *lower star* $(\_{}_*)$ on a state f is defined as $f_* := (f^*)^\dagger = (f^\dagger)^*$

If the following diagram depicts the function $f : A \to B$



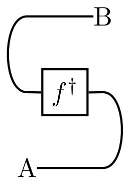
(160)

then the upper star f^* is



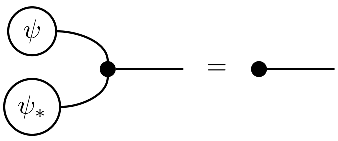
(161)

and the lower star f_* is



(162)

Note that if $f : A \to B$ then $f^* : B \to A$, and $f_* : A \to B$. In other words the lower star is a covariant functor (it doesnt reverse the direction of morphisms) since the upper star and dagger are both contravariant (they do). The upper star is like the transpose operation in linear algebra, and the lower star is like complex conjugation. A state ψ is *unbiased* with respect to a basis structure $\Delta = \{A, \delta, \epsilon\}$ iff $\psi \odot \psi_* = \epsilon^\dagger$:



. (163)

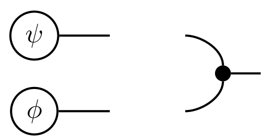
If we consider all of the states in the category together with this basis structure multiplication, then because we can choose $\epsilon^\dagger$ to be an identity under the multiplication $\odot$, we have a monoid (which is a group with some elements which do not have an inverse). If we then pare this down to states that have inverses (equivalently we take the set of unbiased states for Δ) then they form a group with the $\odot$ as multiplication, $\epsilon^\dagger$ as the identity element and the lower star $(\_)_*$ as the operator providing each element with an inverse. This group is known as the **Phase Group**.

5.8 Phase Group for FHilb

For a basis structure on an n-dimensional $\mathcal{H}$ with basis $\{|i\rangle\}$, the basis structure multiplication is

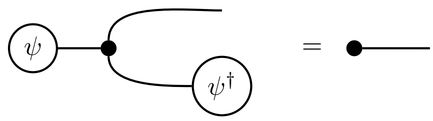
$$|\psi \odot \phi\rangle = \sum_{i=1}^{n} \psi_i \phi_i |i\rangle \tag{164}$$

because we take $\sum_i \psi_i |i\rangle \otimes \sum_j \phi_j |j\rangle$ and act with $\delta^\dagger :: |ii\rangle \mapsto |i\rangle$



. (165)

The unbiased states satisfy



. (166)

Definition A vector $|\psi\rangle$ is unbiased with respect to a basis $\{|a_i\rangle\}_{i=1}^{n}$ iff $\langle a_i|\psi\rangle = \frac{1}{\sqrt{n}}e^{i\theta}$ for some θ in the range $[0, 2\pi]$ (assuming the states are normalised). The important point is that the inner product must be a constant, i.e. independent of the basis vector.

Hence the unbiased states (for a basis structure copying $z\pm$) can be specified uniquely by the relative phase in $|\psi\rangle = |0\rangle + e^{i\phi}|1\rangle$ since all of these states are unbiased with respect to $|0\rangle$ and $|1\rangle$. This means the phase group for **FHilb** is isomorphic to U(1).

5.9 Stab

Definition **FHilb** is the category which has Hilbert Spaces as its Objects and linear maps as its morphisms. The tensor bifunctor is the tensor product and the dagger is the adjoint.

Definition A subcategory is a category with some objects and morphisms removed.

Stab is a subcategory of **FHilb**. To construct the category we remove from Ob(**FHilb**) anything which is not an nth power of a qubit $\mathcal{Q} := \mathbb{C}^2$. We also remove from Hom(**FHilb**) all linear maps that are not single qubit Clifford unitary gates, but leave the linear maps

$$
\begin{aligned}
\delta_{stab} : \mathcal{Q} &\rightarrow \mathcal{Q} \otimes \mathcal{Q} && (167)\\
|0\rangle &\mapsto |00\rangle && (168)\\
|1\rangle &\mapsto |11\rangle && (169)
\end{aligned}
$$

and ϵ_{stab}. Of course we are free to chose from two other possibilities[11] for the eigenstates copied by δ, and within each of those choose from four possibilities for ϵ, so there are 12 basis structures in the theory. These primitive objects, known as generators, are enough to generate the whole theory, that is to say any state or process can be constructed from combining these generators alone. There are a few facts about Stabilizer Quantum Mechanics that reduce the number of generators needed for the theory.

Theorem 5.1. *Any n level clifford unitary gate can be decomposed as two level Clifford unitaries and the CNOT gate.*

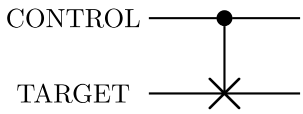
Theorem 5.2. *We can build a CNOT gate using the Hadamard gate which is a two level Clifford unitary.*

A CNOT gate candidate is given in [9] and I prove below that it is indeed the CNOT gate. Once this argument is given, we should be convinced that all of the states and unitary gates of Stabilizer Quantum Mechanics can be constructed from the generators above.

[11] Ordinarily we could choose from a continuum of possibilities but since **Stab** has only a limited number of mutually unbiased bases we are somewhat restricted.

5.10 The CNOT gate for Stab

In Dirac notation, the CNOT is an operation on a bipartite hilbert space which will invert the value of the second qubit if the first qubit is in the state $|1\rangle$. Otherwise it has no effect. The circuit diagram is

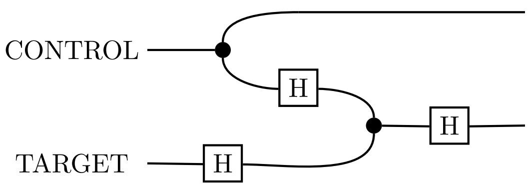


(170)

and if we label the bipartite states with the control qubit first on the left and the target qubit on the right, the following table summarises the CNOT gate:

$$\text{CNOT} :: \begin{cases} |0\rangle|1\rangle & \mapsto & |0\rangle|1\rangle \\ |0\rangle|0\rangle & \mapsto & |0\rangle|0\rangle \\ |1\rangle|0\rangle & \mapsto & |1\rangle|1\rangle \\ |1\rangle|1\rangle & \mapsto & |1\rangle|0\rangle \end{cases} \tag{171}$$

It is suggested [9] that the following picture can achieve an equivalence to the CNOT gate in the pictorial calculus:



(172)

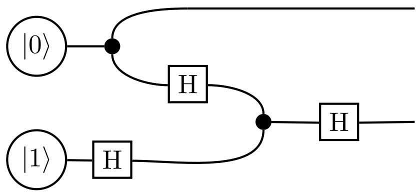
Now we can run each of the four important choices in (171) through this circuit and check the results are as required. The exercise serves as a useful example of how to do calculations using these weird pictures and also highlights the similarity of the pictorial calculus with the familiar circuit diagrams. We require two facts to do computations with this CNOT gate. Firstly we must know the action of H which is involutive:

$$\begin{aligned} H|0\rangle &= |+\rangle \\ H|1\rangle &= |-\rangle \\ H|+\rangle &= |0\rangle \\ H|-\rangle &= |1\rangle \end{aligned} \tag{173}$$

and secondly we must know the action of δ which is included above and also of $\delta^\dagger$ which although a morphism $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ will annihilate certain basis states because δ will only copy eigenstates of the basis structure. In category theoretic language $\delta^\dagger$ is sometimes a morphism to the identity object.

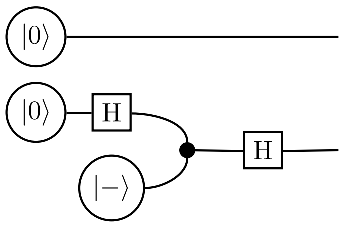
$$\delta^\dagger :: \begin{cases} |00\rangle & \mapsto & |0\rangle \\ |11\rangle & \mapsto & |1\rangle \\ |01\rangle & \mapsto & 1 \\ |10\rangle & \mapsto & 1 \end{cases} \tag{174}$$

Ok now lets input $|0\rangle$ for the control qubit and $|1\rangle$ for the target qubit:



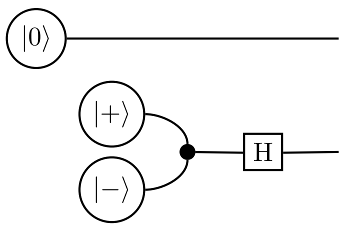
(175)

we can step forward one step by using the rules of the calculus:



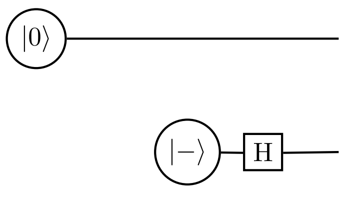
(176)

and again



(177)

and again



(178)

where I have used $\delta^{\dagger} :: ((|0\rangle + |1\rangle)(|0\rangle - |1\rangle)) = \delta^{\dagger} :: (|00\rangle + |10\rangle - |01\rangle - |11\rangle)) \mapsto |-\rangle$ and ignored any global phases and scalars. Finally we get

(179)

as desired. □

The other three cases can be tested in the same fashion.

5.11 Spek

Definition FRel is the category which has Sets as its Objects and relations as its morphisms. The tensor bifunctor is the Cartesian product and the dagger is the relational converse [8]. The identity object is the set with only one element, denoted $\{*\}$. The scalars of the theory are yes/no answers, otherwise known as elements of the Boolean Algebra $\mathbb{B}$.

Spek is a subcategory of **FRel**. If we only consider states of maximal knowledge (pure toy bits), **Spek** is the category of Spekken's toy theory. It inherits the identity object $\{*\}$ from **FRel** , and its only other objects are the four element set IV:= $\{1, 2, 3, 4\}$ and its n-fold Cartesian products IV^n. Allowed morphisms are all permutations on IV, and of course the copying procedure

$$\delta_{spek} : \text{IV} \sim \text{IV} \otimes \text{IV} \tag{180}$$
$$1 \sim \{(1,1),(2,2)\} \tag{181}$$
$$2 \sim \{(1,2),(2,1)\} \tag{182}$$
$$3 \sim \{(3,3),(4,4)\} \tag{183}$$
$$4 \sim \{(3,4),(4,3)\} \tag{184}$$

and deleting procedure ϵ_{spek}. The copying procedure is nicely codified by the following diagram [7]:

		4	3
		3	4
2	1		
1	2		

(185)

Cartesian products are written as ordered pairs (μ, ν) and correspond to a shaded square on a 4×4 grid with coordinate ν, μ in accordance with the convention established by Spekkens [15] and used in Section 3. Mirroring the language used to defined **Stab** above, we can say that this particular choice of δ copies the orthogonal states $(1 \vee 2)$ and $(3 \vee 4)$ and maps other states to correlated (entangled) states.

5.12 Comparison of Stab and Spek

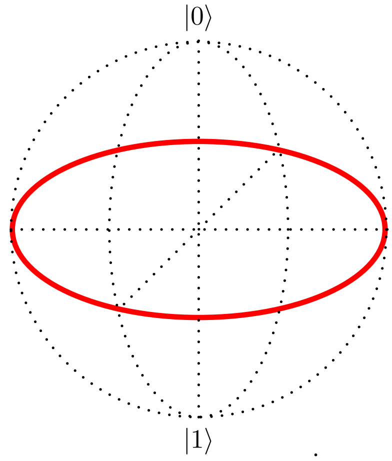
At this stage it looks like Stab and Spek might be the same category. Of course a Hilbert Space is not a strictly a Set, and a linear map is not strictly a relation but *prima facie* there is more than a strong similarity between the Categories if their internal structure is ignored. In fact it seems there is a bijection between states in the theory. For one qubit (elementary system) there are 6 stabilizer (toy bit) states, for two qubits (elementary systems) there are 60 stabilizer (toy crumb) states and furthermore theses states factorize into 24 entangled states and 36 product states. There are 24 morphisms which combine as the group S_4. Both theories exhibit 12 basis structures each with two eigenstates. There are always four states in the phase group for **Stab** and **Spek**, and these four states are the 'other' states that aren't copied by the basis structure chosen. In both theories we are free to choose the identity object: this gives four possible basis structures for each of the three observables. That's twelve basis structures altogether. There is actually a difference, however, in the way that the phase group elements interact with the observable [8] in the two theories. The phase groups are different. It is to category theory's credit that it can quantitatively distinguish the structures of two theories because the theories have incompatible predictions[12]. It is these incompatible predictions that are of interest (i.e. locality versus non-locality) to the programme of classifying quantum phenomena by toy theories. Category theory putatively elucidates which structures of the theories lead to which phenomena. It is stated that the structure of the phase group is precisely the piece of structure which determines if a theory is local or non-local [9, 8].

5.13 Phase Group Structure

The states in the phase group are unbiased with respect to the eigenstates of a given basis structure. Henceforth I assume the basis structure with $z_\pm$ eigenstates, and discuss the structure of the phase group for unrestricted quantum mechanics, Stabilizer quantum mechanics, and Spekkens's Toy Theory. In all cases 'unbiased' will translate into 'equidistant' in the geometrical representation of states for each theory. Question: Which pure states are equidistant in the Bloch sphere from $|0\rangle$ and $|1\rangle$? Answer: A continuous U(1) ring, here shown

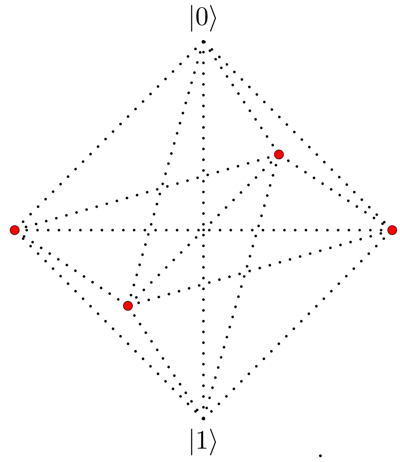
[12]Since the toy theory is explicitly local, it will not violate Bell inequalities.

as a thick red line:



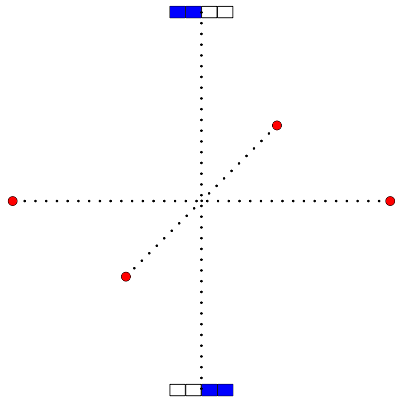
. (186)

Question: Which pure states are equidistant in the stabilizer octahedron from $|0\rangle$ and $|1\rangle$? Answer: the other four observable eigenstates $\{|i\rangle, |i^-\rangle, |+\rangle, |-\rangle\}$ shown as red dots:



. (187)

Question: Which states are equidistant in the toy bit insect from $(1 \vee 2)$ and $(3 \vee 4)$? Answer: the other four observable eigenstates again shown as red dots:



. (188)

It might seem that the phase groups for **Stab** and **Spek** are the same but they are not isomorphic. A bijection may exist but the group multiplication is not preserved under the mapping of the dictionary (59). In other words there is not a homomorphism. Let ξ represent the mapping of the dictionary. In general:

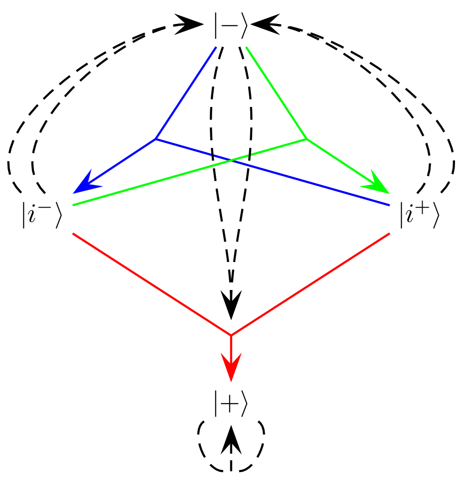
$$\xi(|\psi\rangle \odot |\phi\rangle) \neq \xi(|\psi\rangle) \odot \xi(|\phi\rangle) \tag{189}$$

Elements combine under $\odot$ in a slightly different way: this difference in the group structure of the Phase groups of the two categories can be seen in the diagrams below. If we take the multiplication table for any four element abelian group $\{\alpha, \beta, \gamma, \zeta\}$ then we can ignore multiplications below the diagonal by symmetry (the group is Abelian) and also the trivial identity multiplication. This leaves just three 'squaring' multiplications (i.e. $\mu^2 = \mu \odot \mu$) and three 'combining' multiplications which reveal all of the non-trivial group structure.

$\odot$	α	β	γ	ζ
$\mathbb{I} = \alpha$	#	#	#	#
β	#	β^2	$\beta \odot \gamma$	$\beta \odot \zeta$
γ	#	#	γ^2	$\gamma \odot \zeta$
ζ	#	#	#	ζ^2

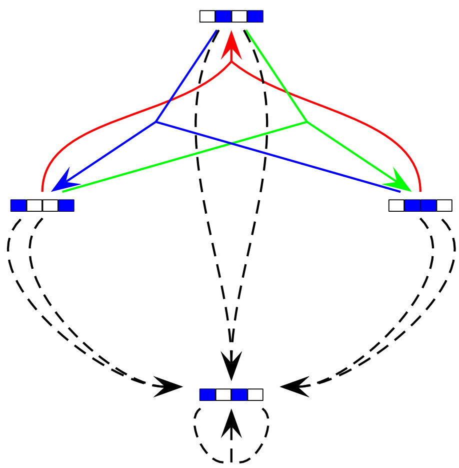
I plot these six group multiplications for the Z_4 phase group of Stab on a two dimensional slice through the stabilizer octahedron, and also include the $\mathbb{I} \odot \mathbb{I} = \mathbb{I}$

mapping for clarity:



(190)

I also plot the six multiplications for the $Z_2 \times Z_2$ phase group of Spek on a two dimensional slice through the toy bit insect again including the $\mathbb{I} \odot \mathbb{I} = \mathbb{I}$ mapping:



(191)

The way to read these diagrams is to interpret each arrow as pointing from two elements in the group to their product under the commutative binary operation $\odot$. The dotted lines point from a group element to it's square under $\odot$. I have kept $\epsilon^\dagger$, the identity object of the phase group, lowermost in these diagrams, as is obvious in the diagram where we see the identity mapping to itself under squaring. Notice the uppermost element squares to the identity in both cases and hence is known as the involutive element. I believe these diagrams to be more informative than multiplication tables: one can immediately spot the difference in group structure[13].

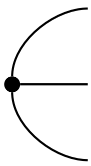
5.14 GHZ state

In this section I hope to show that the non-locality of Stabilizer Quantum Mechanics can be traced to it's Z_4 phase group and that the absence of non-locality in Spekkens's toy theory is due to it's having a $Z_2 \times Z_2$ phase group [9, 7, 8].

Definition The category theory the GHZ state is a morphism $\Psi_\Delta : I \to A \otimes A \otimes A$

$$\Psi_\Delta := (\delta \otimes 1_A) \circ \delta \circ \epsilon^\dagger \tag{192}$$

or pictorially.



(193)

In **Spek** this state is [15]

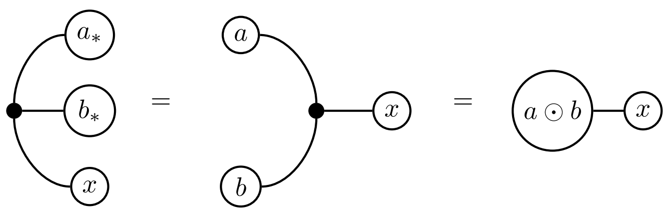
$$(1.1.1) \vee (1.2.2) \vee (2.1.2) \vee (2.2.1) \vee (3.3.3) \vee (3.4.4) \vee (4.3.4) \vee (4.4.3). \tag{194}$$

Definition An *Allowed Triple* (a, b, c) with respect to a tripartite state ψ is one with a nonzero state outcome scalar (e.g. a nozero probability amplitude):

$$(a \otimes b \otimes c)^\dagger \circ \psi \neq 0 \tag{195}$$

Theorem 5.3. *All allowed triples with respect to the GHZ state are of the form* $(a_*, b_*, a \odot b)$ *or equivalently* $(a, b, (a \odot b)_*)$ *, with all other triples 'forbidden' (they have zero state outcome scalars).*

Proof.



(196)

and $x = a \odot b$ is the only solution giving a non-zero outcome scalar [9]. □

13 Any discrete Abelian group could clearly be represented in this way, even for larger groups.

In **Stab** allowed triples correspond to states the have non-zero fidelity with the GHZ state (98). For example $|-\rangle|-\rangle|-\rangle$ is forbidden since:

$$\langle - - -|\mathrm{GHZ}\rangle = 0. \tag{197}$$

The analogue in **Spek** is that $(2 \vee 4).(2 \vee 4).(2 \vee 4)$ is forbidden since it when expanded (when the disjunction is distributed over the conjunction) it's ontic base has a null intersection with the ontic base of GHZ state (194). The usefulness of the $\odot$ multiplication and hence the phase group is now apparent: it enables the construction of triples of states that are compatible with the GHZ state. Allowed triples are compatible with the GHZ state in the sense that a realist is forced to posit hidden variables giving rise to these states in particular: any other state would fail to reproduce the predictions of Quantum Theory in the case of **Stab** and Spekkens's toy theory in the case of **Spek**. Because the phase groups of the theories differ, so do the allowed triples.

5.15 Non-Locality

Bearing in mind Theorem 5.3, we can construct sixteen tripartite states for **Stab** and **Spek** which show the allowed triples for the GHZ state by reading off states from the phase group multiplication table (remembering to take the inverse of the product). I do this by taking the group structure diagrams above and constructing a triple of (origin, origin, inverse of destination) for each arrow. Of course the groups being Abelian means that permutations of the origins of the arrows are also allowed triples, and we must also include the trivial identity multiplications (not shown in the phase group structure diagrams) as allowed triples. The sixteen allowed triples fall into four collections. For **Stab** we have the following triples which happen to all be +1 eigenstates of XXX

$$\begin{matrix} x_+ & x_+ & x_+ \\ x_+ & x_- & x_- \\ x_- & x_+ & x_- \\ x_- & x_- & x_+, \end{matrix} \tag{198}$$

the following which are -1 eigenstates of XYY

$$\begin{matrix} x_+ & y_+ & y_- \\ x_+ & y_- & y_+ \\ x_- & y_+ & y_+ \\ x_- & y_- & y_-, \end{matrix} \tag{199}$$

the following which are -1 eigenstates YXY

$$\begin{matrix} y_+ & x_+ & y_- \\ y_+ & x_- & y_+ \\ y_- & x_+ & y_+ \\ y_- & x_- & y_-, \end{matrix} \tag{200}$$

and the following which are -1 eigenstates of YYX

$$\begin{array}{lll} y_+ & y_+ & x_- \\ y_+ & y_- & x_+ \\ y_- & y_+ & x_+ \\ y_- & y_- & x_-. \end{array} \tag{201}$$

For **Spek** the analogue of a tripartite observable $\sigma\sigma\sigma$ is a measurement distinguishing between two tripartite toy states. For example we can have the observable corresponding to a measurement distinguishing $(1 \vee 3).(1 \vee 3).(1 \vee 3)$ from $(2 \vee 4).(2 \vee 4).(2 \vee 4)$[14]. So, replacing the usual Roman numeral labels I,II with $+, -$ we have the analogue of XXX:

$$\begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \tag{202}$$

of XYX

$$\begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \tag{203}$$

of YXY

$$\begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline\end{array} \tag{204}$$

and of YYX

$$\begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline\end{array} \quad . \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array}. \tag{205}$$

Henceforth these analogues of observables will be treated as such. We have the following allowed triples which happen to all be outcomes associated with the +1 result of XXX

$$\begin{array}{lll} x_+ & x_+ & x_+ \\ x_+ & x_- & x_- \\ x_- & x_+ & x_- \\ x_- & x_- & x_+, \end{array} \tag{206}$$

the following which are outcomes associated with the +1 result of XYY

$$\begin{array}{lll} x_+ & y_+ & y_+ \\ x_+ & y_- & y_- \\ x_- & y_+ & y_- \\ x_- & y_- & y_+, \end{array} \tag{207}$$

the following which are outcomes associated with the +1 result of YXY

$$\begin{array}{lll} y_+ & x_+ & y_+ \\ y_+ & x_- & y_- \\ y_- & x_+ & y_- \\ y_- & x_- & y_+, \end{array} \tag{208}$$

[14]Equivalently $(1.1.1) \vee (1.1.3) \vee (1.3.1) \vee (1.3.3) \vee (3.1.1) \vee (3.1.3) \vee (3.3.1) \vee (3.3.3)$ from $(2.2.2) \vee (2.2.4) \vee (2.4.2) \vee (2.4.4) \vee (4.2.2) \vee (4.2.4) \vee (4.4.2) \vee (4.4.4)$. The two possibilities correspond to two zones of eight shaded cubes with coordinates (z,y,x) in a pictorial representation of tripartite toy states.

and the following which are outcomes associated with the +1 result of YYX

$$\begin{array}{ccc} y_+ & y_+ & x_+ \\ y_+ & y_- & x_- \\ y_- & y_+ & x_- \\ y_- & y_- & x_+ . \end{array} \tag{209}$$

Examine the Mermin table below:

Observable			**Stab**	**Spek**
X	X	X	+	+
X	Y	Y	-	+
Y	X	Y	-	+
Y	Y	X	-	+
+	+	+	!	+

(210)

Using the same four tripartite observables introduced in Section 4.4 we can repeat the GHZ argument in a more abstract way. For a given theory, one can attempt a hidden variable assignment to the observables by populating each row with an allowed triple which is an eigenstate of the appropriate observable. The preceding discussion has excluded forbidden triples, for good reason: they represent states that are incompatible with the GHZ state and therefore useless states to posit as arising from hidden variables. Each triple is made up of eigenstates each with a certain parity. The parity need not be the eigenvalue or any other quantity appearing in the theory, but serves merely as a label for the hidden variable. It might in fact be the argument for a hidden variable function which assigns a completely different value to the state. The parity of each observable is found by multiplying the parity of the eigenstates together. As the parity of each observable is fixed by the allowed triples given above, the row parity can be given automatically and is included on the right of the Mermin table for each theory. The column parity is positive for each column since each observable appears twice in each column. This is true for both theories. The table parity can then be found either by taking the product of the row parities or taking the product of the column parities. Note that we reach a contradiction for **Stab**, but not for **Spek**. This means that of the possible assignments of hidden variables to these observables, no combination of them can be applied without contradiction in **Stab**. Therefore Stabilizer Quantum Mechanics cannot be given a hidden variable interpretation, but Spekkens's toy theory can, as expected[15].

[15] The hidden variable interpretation of Spekkens's toy theory is exactly the values of the ontic states which as we have seen can be given well defined values at all times. These values ultimately give rise to the allowed triples and hence to the failure of the non-locality proof.

6 Conclusion

I have discussed the existence of a no-cloning theorem in both Spekkens's toy theory and in Stabilizer Quantum Mechanics. I take a similar view to Spekkens [15] in that an information theoretic principle is a key conceptual ingredient for this phenomenon. The essence of the Knowledge Balance Principle is that maximal knowledge is incomplete: it seems this conceptual ingredient is highly responsible for many of the quantum like phenomena shared by the toy theory with quantum mechanics. If the toy theory is exquisitely straightforward to understand owing to its classical nature, and the quantum like phenomena are plainly seen to arise from this principle, then perhaps our understanding of Quantum Theory can be improved by installing a similar principle in a reaxiomatization of Quantum Theory. The similarity of the theorem in both the toy bit theory and in Stabilizer Quantum Mechanics strongly suggests that a limit on how much information can be sought from a state should be included in a new set of axioms for Quantum Theory. Whether this limit takes a similar form to the Knowledge Balance Principle, however, is less clear. The principle seems too rigidly discrete to deal with the continuum of states present in unrestricted Quantum Mechanics. Time should be spent on attempting construction of a toy theory which has Spekkens's theory as a restricted sub-theory. Such an expanded theory should be continuous: the canonical set would have to be extended somehow, and the Knowledge Balance Principle reformulated to allow many more states. Only with the construction and evaluation of such a toy theory would a more confident conclusion be drawn about the exact form of any information theoretic axioms for Quantum Theory.

In the tracing of phenomena to conceptual ingredients, differences are as important as similarities. I have discussed the existence of a Non-locality proof in Stabilizer Quantum Mechanics and the absence of such a proof in Spekkens's toy theory. The result is satisfying since a rigorous mathematical foundation is shown to underpin the obvious locality of the toy bit theory. The discrete nature of Stabilizer Quantum Mechanics is not a problem in this consideration, since the GHZ argument can easily be given for **FHilb** by using the U(1) phase group. A generalised proof of which property of the phase group gives rise to non-locality is given in [9], and allows one to predict the locality or non-locality of many more toy theories. This is a very powerful result, and could be instrumental in finding additional non-local toy theories which aid in the research programme which this project suggests. Certainly if a Category Theoretic axiomatization of Quantum Theory is attempted, it should include the phase group: at present this is U(1) but perhaps other continuous phase groups (sharing some attribute with U(1)) also generate non-locality and a more generalised axiom could be given based on the shared attribute.

There are many more similarities between Spekkens's toy theory and Quantum Mechanics, and even between the toy theory and Stabilizer Quantum Mechanics. The toy theory exhibits remote steering, a no broadcasting theorem and dense coding in addition to the no cloning theorem described [15]. There remain many differences aside from the discrete/continuous nature of the states and transformations: coherent binary relations are not precisely analogous to those of Stabilizer Quantum Mechanics, and there are analogues of anti-unitary transformations in the toy theory. Once these similarities and differences have been explored, perhaps with Category Theory, other toy theories can be examined. The similarities and differences in both conceptual ingredients and in predicted phenomena will suggest which toy theories would be useful to examine, or even how to extend or modify current toy theories to most efficiently achieve ingredient isolation.

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